

# One-parameter and multiparameter function classes are intersections of finitely many dyadic classes.\*

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## Abstract

We prove that the class of Muckenhoupt  $A_p$  weights coincides with the intersection of finitely many suitable translates of dyadic  $A_p$ , in both the one-parameter and multiparameter cases, and that the analogous results hold for the reverse Hölder class  $RH_p$ , for doubling measures, and for the space VMO of functions of vanishing mean oscillation. We extend to the multiparameter (product) space BMO of functions of bounded mean oscillation the corresponding one-parameter BMO result due to T. Mei, by means of the Carleson-measure characterization of multiparameter BMO. Our results hold in both the compact and non-compact cases. In addition, we survey several definitions of VMO and prove their equivalences, in the continuous, dyadic, one-parameter and multiparameter cases. We show that the weighted Hardy space  $H^1(\omega)$  is the sum of finitely many suitable translates of dyadic weighted  $H^1(\omega)$ , and that the weighted maximal function is pointwise comparable to the sum of finitely many dyadic weighted maximal functions for suitable translates of the dyadic grid and for each doubling weight  $\omega$ .

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# 1 Introduction

Function spaces and function classes are of considerable interest in harmonic analysis, since (i) the prototypical problem is to establish the boundedness of a singular integral operator from one function space to another, (ii) these operators also act on  $L^p$  spaces weighted by functions in the  $A_p$  or  $RH_p$  function classes, and (iii) the density of the measure on the underlying space is commonly assumed to belong to the class of doubling weights. Dyadic function classes offer a parallel setting in which calculation is often simpler, since one can exploit the geometry of the dyadic intervals. For example, in [JoNi] the John–Nirenberg inequality is proved by establishing a related inequality on certain dyadic cubes arising from a Calderón–Zygmund decomposition. In the current paper we are concerned with a *bridge between continuous and dyadic function classes*.

We consider the *function spaces* BMO, VMO and  $H^1$  (respectively of functions of bounded mean oscillation, functions of vanishing mean oscillation, and the Hardy space), and also the *weight classes* of Muckenhoupt’s  $A_p$  weights, of reverse Hölder  $RH_p$  weights, and of doubling weights. We use the term *function classes* to include both function spaces and weight classes. We show that for each of these function classes, the continuous version can be written as the intersection of finitely many suitable translates of the dyadic version, using two translates in the one-parameter case, and  $2^k$  translates in the multiparameter case. We also show how the norms (or the  $A_p$  and  $RH_p$  constants) from the continuous and dyadic classes are related. Our results hold both in the compact case where our functions are defined on the circle  $\mathbb{T}$  and in the non-compact case of the real line  $\mathbb{R}$ , and for the higher-dimensional one-parameter analogues  $\mathbb{T}^m$  and  $\mathbb{R}^m$ . As corollaries, we extend to the weighted case and to the multiparameter case the result that the Hardy space  $H^1$  is the sum of finitely many suitable translates of the dyadic version of  $H^1$ . Also, we show that the weighted maximal function, where the weight is doubling, is pointwise comparable to the sum of finitely many suitable translates of the dyadic weighted maximal functions.

We note that Mei established these results for the case of one-parameter BMO, for one-parameter unweighted  $H^1$ , and for the one-parameter unweighted maximal function [Mei].

We mention that there is a second type of bridge between continuous and dyadic function classes, via averaging, as developed in the papers [GJ, W, PW, T, PWX]. Namely, a suitable family of functions in the dyadic version of a function class can be converted to a single function that belongs to the continuous version of the same class, via a translation-average (for BMO and VMO) or a geometric-arithmetic average (for  $A_p$ ,  $RH_p$ , and doubling weights). We do not discuss these matters further in the current paper.

What is a “suitable translate”? Denote by  $\mathcal{D}$  the usual grid of dyadic intervals in  $\mathbb{R}$ . Consider the real numbers  $\delta$  that are *far from the dyadic rational numbers*, in the sense that the distance from  $\delta$  to each given dyadic rational  $k/2^n$  is at least some fixed multiple of  $1/2^n$ . That is,

$$\left| \delta - \frac{k}{2^n} \right| \geq \frac{C}{2^n} \quad \text{for all integers } n \text{ and } k, \quad (1.1)$$

where  $C$  is a positive constant that may depend on  $\delta$  but is independent of  $n$  and  $k$ . For

example,  $\delta = 1/3$  is far from the dyadic rationals. The set of all such  $\delta$  is dense in  $\mathbb{R}$  but has measure zero [Mei].

For such a  $\delta$ , let  $\mathcal{D}^\delta$  denote the translate of  $\mathcal{D}$  by  $\delta$ , modified as follows. Small dyadic intervals are simply translated by  $\delta$ . Large dyadic intervals are translated not only by  $\delta$  but also by an additional amount which depends on the scale of the interval. See Section 2 for the precise construction of this collection  $\mathcal{D}^\delta$  of translated dyadic intervals and for a motivating example.

The key proposition in Mei's paper ([Mei, Prop 2.1]) states that if  $\delta$  is far from the dyadic rationals, then for each interval  $Q$  there is an interval  $I$  containing  $Q$ , belonging either to the grid  $\mathcal{D}$  of dyadic intervals or to the grid  $\mathcal{D}^\delta$  of translated dyadic intervals, and whose length  $I$  is no more than  $C|Q|$ , where  $C$  is a constant independent of  $Q$ .

We observe in Proposition 3.4 below that if  $\delta$  is far from the dyadic rationals and if a weight  $\omega$  is both dyadic  $\mathcal{D}$ -doubling and dyadic  $\mathcal{D}^\delta$ -doubling, then for each interval  $Q$ , the average of  $\omega$  on  $|Q|$  is comparable to the average of  $\omega$  on the interval  $I$  guaranteed by [Mei, Prop 2.1], with constants independent of  $Q$ . We use this observation in a crucial way in the proofs of our results for  $A_p$ ,  $RH_p$ ,  $H^1(\omega)$  and weighted maximal functions; see below.

We note that Mei's key proposition is a generalization of the so-called one-third trick. The earliest reference we have found for this idea is in [O, p.339], although it may have been known earlier.

If a nonnegative locally integrable function  $\omega$  belongs to the class of  $A_p$  weights, for some  $p$  with  $1 \leq p \leq \infty$ , then *a fortiori*  $\omega$  belongs to the class  $A_p^d$  of dyadic  $A_p$  weights, for which the defining  $A_p$  condition is only required to hold for dyadic intervals ( $I \in \mathcal{D}$ ), not for all possible intervals. Similarly  $\omega$  belongs to the class  $A_p^\delta$ , for which the  $A_p$  condition is only required to hold for appropriate translates of the dyadic intervals ( $I \in \mathcal{D}^\delta$ ). Thus  $A_p \subset A_p^d \cap A_p^\delta$ .

We show in this paper that if  $\delta$  is far from dyadic rationals in the sense of condition (1.1), then equality holds:  $A_p = A_p^d \cap A_p^\delta$ . Moreover, the constant  $A_p(\omega)$  depends only on the constants  $A_p^d(\omega)$  and  $A_p^\delta(\omega)$ , and vice versa.

We also establish the corresponding result for reverse Hölder weights: if  $\delta$  satisfies condition (1.1) and  $1 \leq p \leq \infty$ , then  $RH_p = RH_p^d \cap RH_p^\delta$ , with the corresponding dependence of the  $RH_p$  constants. Further, the same is true for doubling weights. These results for  $A_p$ , for  $RH_p$  and for doubling weights are collected in Theorem 3.3 for the one-parameter case, and Theorem 3.5 for the multiparameter case with arbitrarily many factors.

Next we point out that T. Mei [Mei] established the corresponding result for BMO on the circle  $\mathbb{T}$ : if  $\delta$  is far from dyadic rationals in the sense of condition (2.1), then  $\text{BMO}(\mathbb{T}) = \text{BMO}_d(\mathbb{T}) \cap \text{BMO}_\delta(\mathbb{T})$ . He extended this result to (one-parameter)  $\mathbb{T}^m$ , showing that  $\text{BMO}(\mathbb{T}^m)$  is the intersection of  $m + 1$  translates of the dyadic version of  $\text{BMO}(\mathbb{T}^m)$ , and similarly to (one-parameter)  $\mathbb{R}^m$ . He notes that John Garnett knew earlier that BMO coincides with the intersection of three translates of dyadic BMO, building on [Gar, p.417].

We give an alternative proof of Mei's BMO result, via the Carleson-measure characterization of BMO together with Mei's key proposition, in order to generalize to the case of multiparameter BMO; see Theorems 4.6 and 4.7.

We also prove the analogous results for one-parameter and multiparameter VMO (Theorem 5.9). In doing so, we survey several definitions of continuous and dyadic VMO in the one- and multiparameter settings, the equivalences among these definitions, and the duality  $\text{VMO}^* = H^1$ .

As a consequence of the results above, we show that the weighted Hardy space  $H^1(\omega)$  is the sum of finitely many suitable translates of dyadic weighted  $H^1(\omega)$ , and show that the weighted maximal function (Hardy–Littlewood maximal function or strong maximal function, as appropriate) is pointwise comparable to the sum of finitely many dyadic weighted maximal functions for suitable translates of the dyadic grid and for every doubling weight  $\omega$  (in particular for weights  $w \in A_p$ ,  $1 \leq p \leq \infty$ ). See Proposition 6.1 and Corollary 6.2 for the one-parameter case and Propositions 6.3, 6.4 and Corollary 6.5 for the multiparameter case.

We make some remarks comparing the compact case (the circle  $\mathbb{T}$ ) with the non-compact case (the real line  $\mathbb{R}$ ). We define the circle to be the unit interval with endpoints identified:  $\mathbb{T} := [0, 1]/(0 \sim 1)$ . Note that non-dyadic intervals  $Q \subset \mathbb{T}$  may wrap around from 1 to 0. First, for the continuous function space BMO and the continuous function classes of  $A_p$ ,  $RH_p$  and doubling weights, there is only a small difference between the compact and non-compact cases: the defining property is assumed to hold only on the intervals contained in  $\mathbb{T}$  as opposed to on all intervals in  $\mathbb{R}$ . Second, for their dyadic versions ( $\text{BMO}_d$ ,  $A_p^d$ ,  $RH_p^d$  and dyadic doubling weights), the same is true, with the additional difference that when considering translations by  $\delta$ , in the compact case ( $\mathbb{T}$ ) it suffices simply to translate each dyadic interval by  $\delta$ , while in the non-compact case  $\mathbb{R}$ , we translate intervals of length larger than 1 not only by the amount  $\delta$  but also by an additional amount that depends on the scale, as mentioned above.

The differences in the case of VMO are more subtle. First, for continuous VMO, in the compact case the definition of the subspace VMO of BMO involves a condition requiring the oscillation of the function to approach zero as the length of the interval goes to zero. In the non-compact case, one must impose two additional conditions controlling the oscillation over large intervals and over intervals that are far from the origin. (With this definition one retains the duality  $\text{VMO}^* = H^1$ .) Second, for the dyadic non-compact case the same three oscillation conditions apply, and also when translating by  $\delta$  we need the additional translations of intervals at large scales, as described in the preceding paragraph. See Section 5 for a detailed discussion of VMO, including references.

For the one-parameter but higher-dimensional cases, where the functions are defined on  $\mathbb{T}^m$  or  $\mathbb{R}^m$ ,  $m \geq 2$ , instead of on  $\mathbb{T}$  or  $\mathbb{R}$ , the same remarks apply, with intervals replaced by cubes.

So much for the one-parameter case. The corresponding remarks apply to the multiparameter case. We give the technical details in the body of the paper.

To reduce the amount of notation required, in the rest of the paper we work on  $\mathbb{R}$  and  $\mathbb{R} \otimes \mathbb{R}$ . However, our results and proofs go through for  $\mathbb{T}$  and  $\mathbb{T} \otimes \mathbb{T}$ , and for  $\mathbb{R}^m$ ,  $\mathbb{R}^{m_1} \otimes \mathbb{R}^{m_2}$ ,  $\mathbb{T}^m$ , and  $\mathbb{T}^{m_1} \otimes \mathbb{T}^{m_2}$ , and also for arbitrarily many factors in the multiparameter setting ( $\mathbb{R}^{m_1} \otimes \dots \otimes \mathbb{R}^{m_k}$  or  $\mathbb{T}^{m_1} \otimes \dots \otimes \mathbb{T}^{m_k}$ ).

The paper is organized as follows. In Section 2 we give the required background on grids of dyadic intervals, their translates, real numbers  $\delta$  that are far from dyadic rationals, and the key proposition in Mei's paper. In Section 3 we define the classes of  $A_p$  weights,  $RH_p$  weights and doubling weights, including the extreme cases  $A_1$ ,  $A_\infty$ ,  $RH_1$  and  $RH_\infty$ . We prove our results for all these classes in both the one-parameter and multiparameter cases. In Section 4 we re-prove Mei's BMO result, and extend it to the multiparameter case. In Section 5, we discuss the definitions of VMO and prove the VMO results. In Section 6, we prove our results for the Hardy space  $H^1$  in both the one- and multiparameter cases, and for the Hardy–Littlewood maximal function and the strong maximal function, and we establish the weighted versions of these results.

## 2 Dyadic intervals and their translates

Let  $\mathcal{D} = \mathcal{D}(\mathbb{R})$  denote the grid of dyadic intervals on  $\mathbb{R}$ :

$$\mathcal{D}(\mathbb{R}) = \bigcup_{n \in \mathbb{Z}} \mathcal{D}_n(\mathbb{R}),$$

where for each  $n \in \mathbb{Z}$ ,

$$\mathcal{D}_n(\mathbb{R}) = \left\{ \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right) \mid k \in \mathbb{Z} \right\}.$$

On the circle  $\mathbb{T} = [0, 1]/(0, 1)$ , the definition of  $\mathcal{D}(\mathbb{T})$  is the same except that the integer  $n$  runs only from 0 to  $\infty$ , and  $k \in \{0, 1, \dots, 2^n - 1\}$ .

For  $\delta \in \mathbb{R}$ , we denote by  $\mathcal{D}^\delta = \mathcal{D}^\delta(\mathbb{T})$  the translate to the right by  $\delta$  of the dyadic grid on the circle  $\mathbb{T}$ , considered modulo 1. Thus

$$\mathcal{D}^\delta(\mathbb{T}) := \{I + \delta \mid I \in \mathcal{D}(\mathbb{T})\} = \left\{ \left[ \left( \frac{k}{2^n} + \delta \right) \bmod 1, \left( \frac{k+1}{2^n} + \delta \right) \bmod 1 \right) \mid k \in \mathbb{Z} \right\}.$$

Finally, on the real line  $\mathbb{R}$ , following Mei, we include additional translates in the definition of the large-scale  $\delta$ -dyadic intervals in  $\mathcal{D}^\delta(\mathbb{R})$ . Specifically,

$$\mathcal{D}^\delta(\mathbb{R}) = \bigcup_{n \in \mathbb{Z}} D_n^\delta(\mathbb{R}),$$

where for  $n \geq 0$ ,

$$D_n^\delta(\mathbb{R}) = \{I + \delta \mid I \in \mathcal{D}_n(\mathbb{R})\},$$

while for  $n < 0$  and  $n$  even,

$$D_n^\delta(\mathbb{R}) = \left\{ \left[ \frac{k}{2^n} + \delta + \sum_{j=(n/2)+1}^0 2^{-2j}, \frac{k+1}{2^n} + \delta + \sum_{j=(n/2)+1}^0 2^{-2j} \right) \mid k \in \mathbb{Z} \right\}.$$

These choices together with the nested property completely determine the collections  $\mathcal{D}_n^\delta(\mathbb{R})$  for  $n < 0$ ,  $n$  odd. Note that we have translated by  $\sum_{j=(n/2)+1}^0 2^{-2j}$ , rather than by Mei's translation  $\sum_{j=n+2}^0 2^{-j}$ .

For example, for  $n = -2$ , the interval  $[0, 4)$  of length  $2^2$  belongs to  $\mathcal{D}_{-2}(\mathbb{R})$  while its translate  $[\delta + 1, \delta + 5)$  belongs to  $\mathcal{D}_{-2}^\delta(\mathbb{R})$ , since  $\sum_{j=0}^0 2^{-2j} = 1$ . Similarly, for  $n = -4$ , the interval  $[0, 16)$  of length  $2^4$  belongs to  $\mathcal{D}_{-4}(\mathbb{R})$  while its translate  $[\delta + 5, \delta + 21)$  belongs to  $\mathcal{D}_{-4}^\delta(\mathbb{R})$ , since  $\sum_{j=-1}^0 2^{-2j} = 5$ .

**Definition 2.1.** A real number  $\delta$  is *far from the dyadic rational numbers* if the distance from  $\delta$  to each given dyadic rational  $k/2^n$  is at least some fixed multiple of  $1/2^n$ . That is, if

$$\left| \delta - \frac{k}{2^n} \right| \geq \frac{C}{2^n} \quad \text{for all integers } n \text{ and } k,$$

where  $C$  is a positive constant that may depend on  $\delta$  but is independent of  $n$  and  $k$ . Equivalently, the *relative distance*  $d(\delta)$  from  $\delta$  to the set of dyadic rational numbers is positive:

$$d(\delta) := \inf \left\{ 2^n \left| \delta - \frac{k}{2^n} \right| \mid n \in \mathbb{Z}, k \in \mathbb{Z} \right\} > 0. \quad (2.1)$$

For example,  $\delta = 1/3$  is far from the dyadic rationals since  $d(1/3) = 1/3 > 0$ . The set of all such  $\delta$  is dense in  $\mathbb{R}$  but has measure zero. Note that  $d(\delta + 1) = d(\delta)$  for all  $\delta \in \mathbb{R}$ .

The key tool in Mei's paper is the following proposition.

**Proposition 2.2** (Prop 2.1 [Mei]). *Suppose  $\delta \in (0, 1)$  is far from dyadic rationals, in the sense of condition (2.1). Then there is a constant  $C(\delta)$  such that for each interval  $Q$  in  $\mathbb{R}$ , there is an interval  $I$  in  $\mathbb{R}$  such that*

- (i)  $Q \subset I$ ,
- (ii)  $|I| \leq C(\delta)|Q|$ , and
- (iii)  $I \in \mathcal{D}$  or  $I \in \mathcal{D}^\delta$ .

The constant  $C(\delta)$  can be taken to be  $C(\delta) = 2/d(\delta)$ .

Mei's proposition is stated on the circle  $\mathbb{T}$  identified with  $(0, 2\pi]$ , with condition (2.1) replaced by

$$d(\delta) := \inf \{ 2^n |\delta - k2^{-n}| \mid n \geq 0, k \in \mathbb{Z} \} > 0, \quad (2.2)$$

with  $0 < \delta < 1$  and with the filtrations  $\mathcal{D}(\mathbb{T})$  and  $\mathcal{D}^\delta(\mathbb{T})$ . For completeness, we give a proof, stated on the real line and following Mei's proof.

*Proof of Proposition 2.2.* Fix an interval  $Q$  in  $\mathbb{R}$ . There exists an integer  $n$  such that  $d(\delta)2^{-n-1} \leq |Q| < d(\delta)2^{-n}$ . Now we set  $A_n = \{k \cdot 2^n \mid k \in \mathbb{Z}\}$  for every fixed  $n$  and

- (1)  $A_n^\delta = \{\delta + k \cdot 2^n \mid k \in \mathbb{Z}\}$  for  $n \geq 0$ ,
- (2)  $A_n^\delta = \{\delta + \sum_{j=(n/2)+1}^0 2^{-2j} + k \cdot 2^n \mid k \in \mathbb{Z}\}$  for  $n < 0$ ,  $n$  even, and
- (3)  $A_n^\delta = \{\delta + \sum_{j=(n-1)/2+1}^0 2^{-2j} + k \cdot 2^n \mid k \in \mathbb{Z}\}$  for  $n < 0$ ,  $n$  odd.

Note that for any two different points  $a, b \in A_n \cup A_n^\delta$ , we have  $|a - b| \geq d(\delta)2^{-n} > |Q|$ . Thus, there is at most one element of  $A_n \cup A_n^\delta$  belongs to  $Q$ . Hence,  $Q \cap A_n = \emptyset$  or  $Q \cap A_n^\delta = \emptyset$ . Therefore,  $I$  must be contained in some dyadic interval  $I \in \mathcal{D}$  or  $I \in \mathcal{D}^\delta$  and  $|I| = 2^{-n} \leq (2/d(\delta))|Q|$ .  $\square$

The corresponding result holds for intervals  $I'$  contained in  $Q$ .

**Proposition 2.3.** *Suppose  $\delta \in \mathbb{R}$  is far from dyadic rationals, in the sense of condition (2.1). Then there is a constant  $C'(\delta)$  such that for each interval  $Q$  in  $\mathbb{R}$ , there is an interval  $I'$  in  $\mathbb{R}$  such that*

- (i)  $Q \supset I'$ ,
- (ii)  $|I'| \geq C'(\delta)|Q|$ , and
- (iii)  $I' \in \mathcal{D}$  or  $I' \in \mathcal{D}^\delta$ .

*Proof.* For every interval  $Q$  in  $\mathbb{R}$ , take the integer  $n$  such that  $2^{-n} \leq |Q| < 2^{-n+1}$ . Then, there exists an interval  $I$  in  $\mathcal{D}_{-n-1}$  or  $\mathcal{D}_{-n-1}^\delta$  such that  $I \subset Q$ . Hence the Proposition holds.  $\square$

**Example 2.4.** Here is an example that illustrates the difference between  $\mathbb{R}$  and  $\mathbb{T}$  with regard to translations, and the need in the definition above of the  $\delta$ -dyadic intervals in  $\mathcal{D}^\delta(\mathbb{R})$  for the global translations at certain scales. On the real line  $\mathbb{R}$ , take the usual collection  $\mathcal{D}$  of dyadic intervals, take any positive  $\delta$  and let  $\mathcal{D}^{\delta,\#}$  denote the translation to the right by  $\delta$  of the dyadic grid  $\mathcal{D}$ , so that  $I^\# \in \mathcal{D}^{\delta,\#}$  if and only if  $I^\# = I + \delta$  for some  $I \in \mathcal{D}$ . Let  $Q$  be an interval containing both 0 and  $\delta$  in its interior. Then there is no interval  $I$  in  $\mathcal{D}$  or  $\mathcal{D}^{\delta,\#}$  satisfying property (i), namely  $Q \subset I$ , of Proposition 2.2, since dyadic intervals do not have 0 as an interior point and intervals in  $\mathcal{D}^{\delta,\#}$  do not have  $\delta$  as an interior point.

On the circle  $\mathbb{T}$  viewed as  $[0, 1]$  with the endpoints identified, however, the situation is different. Take  $\delta > 0$  such that  $d(\delta) > 0$ . It follows from the definition of  $\delta$ , by taking  $k = n = 0$ , that  $d(\delta) \leq \delta$ . Let  $Q$  be an interval in  $\mathbb{T}$  containing 0 and  $\delta$  as interior points. Then  $|Q| \geq \delta$ . It follows that the properties asserted in Proposition 2.2 hold for the choice  $I = \mathbb{T}$ , since  $Q \subset \mathbb{T}$ ,  $\mathbb{T} \in \mathcal{D}$ , and  $|\mathbb{T}| = 1 \leq |Q|/\delta$  so that

$$\frac{|\mathbb{T}|}{|I_{t,y}|} \leq \frac{1}{\delta} \leq \frac{1}{d(\delta)} \leq \frac{2}{d(\delta)} = C(\delta).$$

Mei's use of the *translates* of the usual dyadic intervals at scale  $2^{-n}$  for all even  $n < 0$  [Mei, Remark 7] ensures that the conclusion of Proposition 3.4 does hold for the intervals  $Q \subset \mathbb{R}$  in this example.

For a function  $\omega$  that is both dyadic  $\mathcal{D}$ -doubling and dyadic  $\mathcal{D}^\delta$ -doubling, where  $\delta$  is far from dyadic rationals, the average value of  $\omega$  over an interval  $Q$  is comparable to the average value of  $\omega$  over the interval  $I$  guaranteed by Proposition 2.2. See Proposition 3.4 below.

### 3 Intersections of dyadic classes of weights

#### 3.1 One-parameter results for $A_p$ and $RH_p$

We prove that  $A_p$  is the intersection of two suitable translates of dyadic  $A_p$  for  $1 \leq p \leq \infty$ , and that the corresponding results hold for  $RH_p$ ,  $1 \leq p \leq \infty$  and for doubling weights.

As usual, by *doubling weight* we mean a nonnegative locally integrable function  $\omega$  on  $\mathbb{R}$  such that  $\omega(\tilde{Q}) \leq C\omega(Q)$  with a positive constant  $C$  independent of  $Q$ , where the double  $\tilde{Q}$  of  $Q$  is the interval with the same midpoint as  $Q$  and twice the length of  $Q$ . Similarly, a *dyadic doubling weight* satisfies the corresponding condition  $\omega(\tilde{I}) \leq C\omega(I)$ , where  $\tilde{I}$  is the dyadic parent of  $I$ .

The  $A_p$  weights were identified by Muckenhoupt [M] as the weights  $\omega$  for which the Hardy–Littlewood maximal function is bounded from  $L^p(d\mu)$  to itself, where  $d\mu = \omega(x) dx$ . Here we define the classes  $A_p$  and  $RH_p$  in the one-parameter setting. We delay the corresponding definitions, statements and proofs for the multiparameter setting until Subsection 3.2.

We use the notation  $\int_E f := \frac{1}{|E|} \int_E f$ .

**Definition 3.1.** Let  $\omega(x)$  be a nonnegative locally integrable function on  $\mathbb{R}$ . For real  $p$  with  $1 < p < \infty$ , we say  $\omega$  is an  $A_p$  weight, written  $\omega \in A_p$ , if

$$A_p(\omega) := \sup_Q \left( \int_Q \omega \right) \left( \int_Q \left( \frac{1}{\omega} \right)^{1/(p-1)} \right)^{p-1} < \infty.$$

For  $p = 1$ , we say  $\omega$  is an  $A_1$  weight, written  $\omega \in A_1$ , if

$$A_1(\omega) := \sup_Q \left( \int_Q \omega \right) \left( \frac{1}{\operatorname{ess\,inf}_{x \in Q} \omega(x)} \right) < \infty.$$

For  $p = \infty$ , we say  $\omega$  is an  $A$ -infinity weight, written  $\omega \in A_\infty$ , if

$$A_\infty(\omega) := \sup_Q \left( \int_Q \omega \right) \exp \left( \int_Q \log \left( \frac{1}{\omega} \right) \right) < \infty.$$

Here the suprema are taken over all intervals  $Q \subset \mathbb{R}$ . The quantity  $A_p(\omega)$  is called the  $A_p$  constant of  $\omega$ .

The *dyadic  $A_p$  classes*  $A_p^d$  for  $1 \leq p \leq \infty$  are defined analogously, with the suprema  $A_p^d(\omega)$  being taken over only the dyadic intervals  $I \subset \mathbb{R}$ .

**Definition 3.2.** Let  $\omega(x)$  be a nonnegative locally integrable function on  $\mathbb{R}$ . For real  $p$  with  $1 < p < \infty$ , we say  $\omega$  is a *reverse-Hölder- $p$  weight*, written  $\omega \in RH_p$  or  $\omega \in B_p$ , if

$$RH_p(\omega) := \sup_Q \left( \int_Q \omega^p \right)^{1/p} \left( \int_Q \omega \right)^{-1} < \infty.$$



For  $p = 1$ , we say  $\omega$  is a *reverse-Hölder-1 weight*, written  $\omega \in RH_1$  or  $\omega \in B_1$ , if

$$RH_1(\omega) := \sup_Q \int_Q \left( \frac{\omega}{f_Q \omega} \log \frac{\omega}{f_Q \omega} \right) < \infty.$$

For  $p = \infty$ , we say  $\omega$  is a *reverse-Hölder-infinity weight*, written  $\omega \in RH_\infty$  or  $\omega \in B_\infty$ , if

$$RH_\infty(\omega) := \sup_Q \left( \operatorname{ess\,sup}_{x \in Q} \omega \right) \left( \int_Q \omega \right)^{-1} < \infty.$$

Here the suprema are taken over all intervals  $Q \subset \mathbb{R}$ . The quantity  $RH_p(\omega)$  is called the  *$RH_p$  constant* of  $\omega$ .

For  $1 \leq p \leq \infty$ , we say  $\omega$  is a *dyadic reverse-Hölder- $p$  weight*, written  $\omega \in RH_p^d$  or  $\omega \in B_p^d$ , if

- (i) the analogous condition  $RH_p^d(\omega) < \infty$  holds with the supremum being taken over only the dyadic intervals  $I \subset \mathbb{R}$ , and
- (ii) in addition  $\omega$  is a dyadic doubling weight.

We define the  *$RH_p^d$  constant*  $RH_p^d(\omega)$  of  $\omega$  to be the larger of this dyadic supremum and the dyadic doubling constant.

The  $A_p$  inequality (or the  $RH_p$  inequality) implies that the weight  $\omega$  is doubling, and the dyadic  $A_p$  inequality implies that  $\omega$  is dyadic doubling. However, the dyadic  $RH_p$  inequality does not imply that  $\omega$  is dyadic doubling, which is why the dyadic doubling assumption is needed in the definition of  $RH_p^d$ .

We define the  $\delta$ -dyadic classes  $A_p^\delta$  and  $RH_p^\delta$  similarly, using the collection  $\mathcal{D}^\delta$  of intervals defined at the start of Section 2.

It is shown in [BR] that a weight  $\omega$  belongs to  $A_\infty$  if and only if  $\omega$  belongs to  $RH_1$ . Thus,  $RH_1 = A_\infty$  as sets. Moreover, the constants are related by

$$\frac{1}{e} RH_1(\omega) \leq A_\infty(\omega) \leq C \frac{e^{e^{RH_1(\omega)}}}{e^{RH_1(\omega)}},$$

where  $C$  is independent of  $RH_1(\omega)$ . The constant  $1/e$  is sharp, and the right-hand inequality is sharp in  $RH_1(\omega)$ . The same proofs go through for the dyadic case.

We note that  $A_\infty$  is the union of the  $A_p$  classes, which are nested and increasing as  $p \rightarrow \infty$ , and also  $RH_1$  is the union of the  $RH_p$  classes, which are nested and decreasing as  $p \rightarrow \infty$ . Thus,

$$A_\infty = \bigcup_{1 \leq p < \infty} A_p = \bigcup_{1 < p \leq \infty} RH_p = RH_1.$$

See for example [Gar], [Gra] or [GCRF] for the theory and history of  $A_p$  weights,  $1 \leq p \leq \infty$ , and  $RH_p$  weights for  $1 < p < \infty$ . The class  $RH_\infty$  was defined in [CN], and the class  $RH_1$  was defined in [BR] and, via an equivalent definition, in [HP].

**Theorem 3.3.** *Suppose  $\delta \in \mathbb{R}$  is far from dyadic rationals, in the sense that condition (2.1) holds. Then the following assertions hold:*

- (a)  $\omega$  is a doubling weight if and only if  $\omega$  is dyadic doubling with respect to  $\mathcal{D}$  and with respect to  $\mathcal{D}^\delta$ .
- (b)  $A_p = A_p^d \cap A_p^\delta$ , for each  $p$  with  $1 \leq p \leq \infty$ .
- (c)  $RH_p = RH_p^d \cap RH_p^\delta$ , for each  $p$  with  $1 \leq p \leq \infty$ .

*Bounds for the constants are given in the proof below. In particular,  $A_p(\omega)$  depends only on  $A_p^d(\omega)$  and  $A_p^\delta(\omega)$ , and vice versa, and similarly for the other cases.*

It is immediate that if  $\omega$  is doubling then it is dyadic doubling with respect to both  $\mathcal{D}$  and  $\mathcal{D}^\delta$ , and similarly if  $\omega$  lies in  $A_p$  (respectively  $RH_p$ ) then  $\omega$  lies in both  $A_p^d$  and  $A_p^\delta$  (respectively  $RH_p^d$  and  $RH_p^\delta$ ). In proving the other direction, the key point is that the average of  $\omega$  over an interval  $Q$  is comparable to the average of  $\omega$  over the interval  $I \supset Q$  guaranteed by Mei's proposition. More precisely, we have the following proposition.

**Proposition 3.4.** *If  $\delta$  is far from the dyadic rationals and  $\omega$  is both dyadic  $\mathcal{D}$ -doubling and dyadic  $\mathcal{D}^\delta$ -doubling with constant  $C_{\text{dy}}$ , then given an interval  $Q$  and given the dyadic interval  $I$  guaranteed by Proposition 2.2, we have*

$$(C_{\text{dy}})^{-\log_2(4C(\delta))} \int_I \omega \leq \int_Q \omega \leq C(\delta) \int_I \omega. \quad (3.1)$$

*Proof.* Let  $N$  be the unique integer such that  $2^{N-1} < C(\delta) \leq 2^N$ . Then  $N+1 < \log_2(4C(\delta))$ , and

$$\frac{|I|}{2^{N+1}} \leq \frac{|I|}{2C(\delta)} \leq \frac{|Q|}{2}.$$

Therefore, considering the  $2^{N+1}$  subintervals  $J$  of  $I$  of length  $|J| = |I|/2^{N+1}$  ( $J$  dyadic if  $I \in \mathcal{D}$ ,  $\delta$ -dyadic if  $I \in \mathcal{D}^\delta$ ), we can see that one of these intervals  $J$  must be completely contained in  $Q$ . For this  $J$ , we have

$$\int_I \omega = \frac{1}{|I|} \int_I \omega \leq (C_{\text{dy}})^{N+1} \frac{1}{|I|} \int_J \omega \leq (C_{\text{dy}})^{\log_2(4C(\delta))} \frac{1}{|Q|} \int_Q \omega = (C_{\text{dy}})^{\log_2(4C(\delta))} \int_Q \omega.$$

Moreover, since  $Q \subset I$ ,  $|I| \leq C(\delta)|Q|$ , and  $\omega \geq 0$ , we have

$$\int_Q \omega = \frac{1}{|Q|} \int_Q \omega \leq \frac{C(\delta)}{|I|} \int_I \omega = C(\delta) \int_I \omega. \quad \square$$

We note that if  $\omega$  belongs to  $A_p^d$  or  $RH_p^d$  then  $\omega$  is dyadic  $\mathcal{D}$ -doubling. Similarly  $A_p^\delta$  or  $RH_p^\delta$  functions are dyadic  $\mathcal{D}^\delta$ -doubling. Thus functions in  $A_p^d \cap A_p^\delta$  or in  $RH_p^d \cap RH_p^\delta$  have the comparability property (3.1), by Proposition 3.4.

*Proof of Theorem 3.3.* We need only prove the reverse inclusions ( $\supset$ ).

(a) Suppose  $\omega$  is dyadic doubling with respect to both  $\mathcal{D}$  and  $\mathcal{D}^\delta$ . We show that  $\omega$  is doubling. Take an interval  $Q$  in  $\mathbb{R}$ . The double  $\tilde{Q}$  of  $Q$  is the interval  $\tilde{Q}$  that has the same midpoint as  $Q$  and twice the length:  $|\tilde{Q}| = 2|Q|$ . Let  $I$  be an interval of the type guaranteed by Proposition 2.2 applied to  $\tilde{Q}$ . Take  $N$  such that  $2^{N-1} \leq C(\delta) < 2^N$ . Then we have  $|I| \leq 2^N |\tilde{Q}|$ .

Consider the dyadic subintervals  $J$  of  $I$  at scale  $|J| = 2^{-N-2}|I|$ . These dyadic subintervals  $J$  satisfy  $|J| \leq |Q|/2$ , which implies that  $Q$  contains at least one such dyadic subinterval, denoted by  $J$ . Now we have

$$\int_{\tilde{Q}} \omega \leq \int_I \omega \leq C_{\text{dy}}^{N+2} \int_J \omega \leq C_{\text{dy}}^{N+2} \int_Q \omega \leq C_{\text{dy}}^{\log_2(2^3 C(\delta))} \int_Q \omega.$$

Thus  $\omega$  is doubling, with doubling constant at most  $C_{\text{dy}}^{\log_2(2^3 C(\delta))}$ .

(b) Suppose  $\omega$  belongs to both  $A_p^d$  and  $A_p^\delta$ , for some  $p$  with  $1 < p < \infty$ . Take an interval  $Q$  in  $\mathbb{R}$ . Let  $I$  be the interval guaranteed by Proposition 2.2 applied to  $Q$ . Then from Proposition 3.4, we have

$$\int_Q \omega \leq C(\delta) \int_I \omega.$$

Similarly, since  $\omega \in A_p^d \cap A_p^\delta$  implies that  $\omega^{-\frac{1}{p-1}}$  belongs to  $A_{p'}^d \cap A_{p'}^\delta$  where  $p'$  is the conjugate index of  $p$ , we have

$$\int_Q \left( \frac{1}{\omega} \right)^{\frac{1}{p-1}} \leq C(\delta) \int_I \left( \frac{1}{\omega} \right)^{\frac{1}{p-1}}.$$

Therefore

$$\left( \int_Q \omega \right) \left( \int_Q \left( \frac{1}{\omega} \right)^{\frac{1}{p-1}} \right)^{p-1} \leq C(\delta)^p \left( \int_I \omega \right) \left( \int_I \left( \frac{1}{\omega} \right)^{\frac{1}{p-1}} \right)^{p-1} \leq C(\delta)^p \max\{A_p^d(\omega), A_p^\delta(\omega)\}.$$

Thus  $\omega$  lies in  $A_p$ , with  $A_p$  constant

$$A_p(\omega) \leq C(\delta)^p \max\{A_p^d(\omega), A_p^\delta(\omega)\}.$$

For  $p = 1$ , suppose  $\omega$  belongs to both  $A_1^d$  and  $A_1^\delta$ , and let  $V_1^d = \max\{A_1^d(\omega), A_1^\delta(\omega)\}$ . Take  $Q$ ,  $I$ , and  $N$  as above. By Proposition 3.4 we see that

$$\int_Q \omega \leq C(\delta) \int_I \omega \leq C(\delta) V_1^d \operatorname{ess\,inf}_{x \in I} \omega(x) \leq C(\delta) V_1^d \operatorname{ess\,inf}_{x \in Q} \omega(x).$$

Thus  $\omega \in A_1$  with

$$A_1(\omega) \leq C(\delta) \max\{A_1^d(\omega), A_1^\delta(\omega)\}.$$

For  $p = \infty$ , if  $\omega \in A_\infty^d \cap A_\infty^\delta$ , then there exist  $p_1$  and  $p_2$  such that  $\omega \in A_{p_1}^d$  and  $\omega \in A_{p_2}^\delta$ . Let  $p = \max\{p_1, p_2\}$ . Then  $\omega \in A_p^d \cap A_p^\delta$ , so by the cases  $1 \leq p < \infty$  above, we have  $\omega \in A_p \subset A_\infty$ .

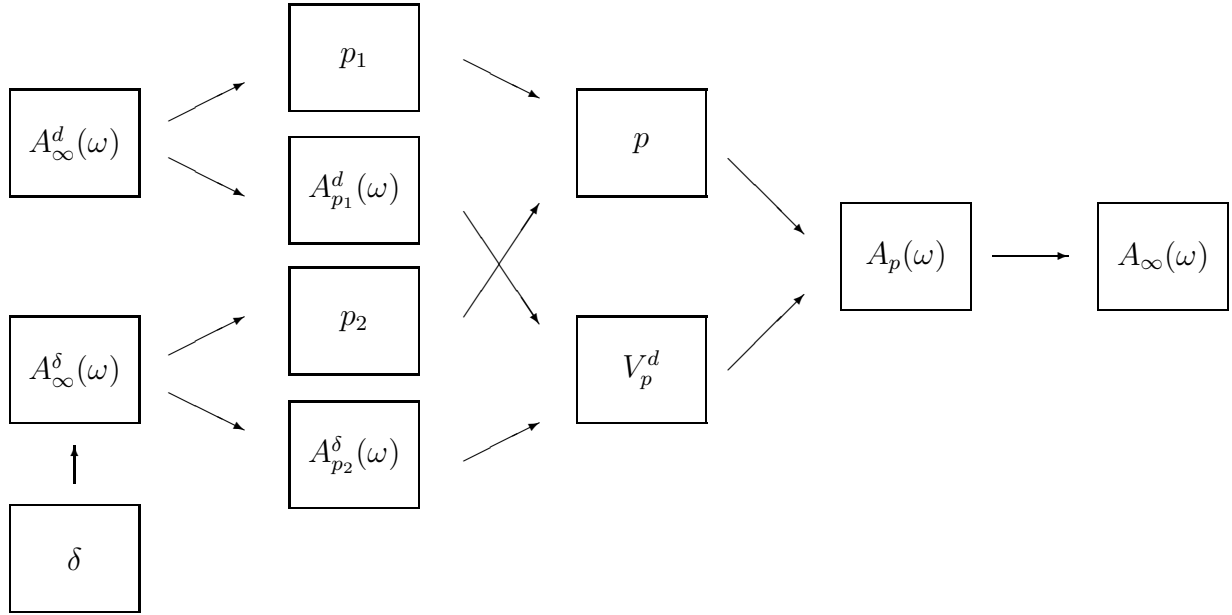


Figure 1: Dependence of the constants in the proof of Theorem 1, for the case of  $A_\infty$ . Here  $V_p^d := \max\{A_p^d(\omega), A_p^\delta(\omega)\}$ .

The dependence of the constants is shown in Figure 1. In particular  $A_\infty(\omega)$  depends only on  $\max\{A_\infty^d(\omega), A_\infty^\delta(\omega)\}$ .

(c) Suppose  $\omega$  belongs to  $RH_p^d \cap RH_p^\delta$ , for some  $p$  with  $1 < p < \infty$ . Then  $\omega$  is both dyadic  $\mathcal{D}$ -doubling and dyadic  $\mathcal{D}^\delta$ -doubling with doubling constant  $C_{\text{dy}}$ . Moreover,  $\omega^p$  is also both dyadic  $\mathcal{D}$ -doubling and dyadic  $\mathcal{D}^\delta$ -doubling, since for each  $I \in \mathcal{D}$  or  $\mathcal{D}^\delta$ ,

$$\begin{aligned} \left( \int_{\tilde{I}} \omega^p \right)^{1/p} &\leq RH_p(\omega) \int_{\tilde{I}} \omega = \frac{RH_p(\omega)}{2} \frac{1}{|I|} \int_{\tilde{I}} \omega \leq \frac{RH_p(\omega)}{2} C_{\text{dy}} \frac{1}{|I|} \int_I \omega \\ &\leq \frac{RH_p(\omega)}{2} C_{\text{dy}} \left( \int_I \omega^p \right)^{1/p}, \end{aligned}$$

which yields that

$$\int_{\tilde{I}} \omega^p \leq 2 \left( \frac{RH_p(\omega)}{2} C_{\text{dy}} \right)^p \int_I \omega^p,$$

where  $\tilde{I}$  is the dyadic parent of  $I$ . That is,  $\omega^p$  is also both dyadic  $\mathcal{D}$ -doubling and dyadic  $\mathcal{D}^\delta$ -doubling with constant  $2^{1-p} (RH_p(\omega) C_{\text{dy}})^p$ .

Now take  $Q$ ,  $I$ , and  $N$  as in part (b). Since  $\omega$  and  $\omega^p$  are both dyadic  $\mathcal{D}$ -doubling and

dyadic  $\mathcal{D}^\delta$ -doubling, Proposition 3.4 implies that

$$\left(\int_Q \omega^p\right)^{1/p} \leq C(\delta)^{1/p} \left(\int_I \omega^p\right)^{1/p} \quad \text{and} \quad \int_I \omega \leq (C_{\text{dy}})^{\log_2(4C(\delta))} \int_Q \omega.$$

Thus

$$\left(\int_Q \omega^p\right)^{1/p} \left(\int_I \omega\right)^{-1} \leq C(\delta)^{1/p} (C_{\text{dy}})^{\log_2(4C(\delta))} \left(\int_I \omega^p\right)^{1/p} \left(\int_I \omega\right)^{-1}.$$

So  $\omega$  belongs to  $RH_p$ , and

$$RH_p(\omega) \leq C(\delta)^{1/p} (C_{\text{dy}})^{\log_2(4C(\delta))} \max\{RH_p^d(\omega), RH_p^\delta(\omega)\}.$$

For  $p = 1$ , suppose  $\omega$  belongs to both  $RH_1^d$  and  $RH_1^\delta$ . By the comment after Definition 3.2,  $\omega$  belongs to both  $A_\infty^d$  and  $A_\infty^\delta$ , with constants depending only on  $RH_1^d(\omega)$  and  $RH_1^\delta(\omega)$ . So by part (b) above,  $\omega$  belongs to  $A_\infty$ , and by the same comment,  $\omega$  belongs to  $RH_1$ , with constant  $RH_1(\omega)$  depending only on  $RH_1^d(\omega)$  and  $RH_1^\delta(\omega)$ .

For  $p = \infty$ , suppose  $\omega$  belongs to both  $RH_\infty^d$  and  $RH_\infty^\delta$ , and let  $V_\infty^d = \max\{RH_\infty^d(\omega), RH_\infty^\delta(\omega)\}$ . Take  $Q$ ,  $I$ , and  $N$  as above. Then

$$\text{ess sup}_{x \in Q} \omega(x) \leq \text{ess sup}_{x \in I} \omega(x) \leq V_\infty^d \int_I \omega \leq V_\infty^d (C_{\text{dy}})^{\log_2(4C(\delta))} \int_Q \omega.$$

Thus  $\omega$  belongs to  $RH_\infty$  and

$$RH_\infty(\omega) \leq (C_{\text{dy}})^{\log_2(4C(\delta))} \max\{RH_\infty^d(\omega), RH_\infty^\delta(\omega)\}. \quad \square$$

### 3.2 Multiparameter results for $A_p$ and $RH_p$

We extend the above results for  $A_p(\mathbb{R})$  and  $RH_p(\mathbb{R})$  to the multiparameter  $(\mathbb{R} \otimes \cdots \otimes \mathbb{R})$  setting. For ease of notation, we write the statements and proofs for the product space  $\mathbb{R} \otimes \mathbb{R}$  of two factors. The same proofs go through for arbitrarily many factors.

As noted in [PWX], the theory of product weights was developed by K.-C. Lin in his thesis [Lin], while the dyadic theory was developed in Buckley's paper [Buc]. The product  $A_p$  and  $RH_p$  weights and the product doubling weights, and their dyadic analogues, are defined exactly as in Definitions 3.1–3.2, with intervals in  $\mathbb{R}$  being replaced by rectangles in  $\mathbb{R} \otimes \mathbb{R}$ . It follows that a product weight belongs to  $A_p(\mathbb{R} \otimes \mathbb{R})$  if and only if it belongs to  $A_p(\mathbb{R})$  in each variable separately.

To be precise,  $\omega \in A_p(\mathbb{R} \otimes \mathbb{R})$  if and only if  $\omega(\cdot, y) \in A_p(\mathbb{R})$  uniformly for a.e.  $y \in \mathbb{R}$  and  $\omega(x, \cdot) \in A_p(\mathbb{R})$  uniformly for a.e.  $x \in \mathbb{R}$ . In one direction this is a consequence of the Lebesgue Differentiation Theorem, letting one side of the rectangle shrink to a point. The converse uses the equivalence between  $\omega \in A_p(\mathbb{R} \otimes \mathbb{R})$  and maximal inequality of the strong maximal function [Ste, p.83]. Further, the  $A_p(\mathbb{R} \otimes \mathbb{R})$  constant depends only on the two  $A_p(\mathbb{R})$  constants, and vice versa.

The analogous characterizations in terms of the separate variables hold for product  $RH_p$  weights and for product doubling weights, and for the dyadic product  $A_p$ ,  $RH_p$ , and doubling weights.

We denote by  $A_p^{d,d} = A_p^{d,d}(\mathbb{R} \otimes \mathbb{R})$  the class of strong dyadic weights, meaning the weights  $\omega(x, y)$  such that

- (i) for a.e. fixed  $y$ ,  $\omega(\cdot, y)$  lies in  $A_p^d(\mathbb{R})$ , and
- (ii) for a.e. fixed  $x$ ,  $\omega(x, \cdot)$  lies in  $A_p^d(\mathbb{R})$ ,

with uniform  $A_p^d(\mathbb{R})$  constants. The class  $A_p^{d,\delta}$  is the same as  $A_p^{d,d}$  except that  $\omega(x, \cdot)$  lies in  $A_p^\delta(\mathbb{R})$ , using in the second variable the translated dyadic grid  $\mathcal{D}^\delta$ . Similarly for  $A_p^{\delta,d}$  and  $A_p^{\delta,\delta}$ , and for the corresponding variations of  $RH_p^{d,d}$ .

**Theorem 3.5.** *Suppose  $\delta \in \mathbb{R}$  is far from dyadic rationals, in the sense that (2.1) holds. Then the following assertions hold:*

- (a) *A weight  $\omega(x, y)$  is a product doubling weight if and only if  $\omega$  is dyadic doubling with respect to each of  $\mathcal{D} \times \mathcal{D}$ ,  $\mathcal{D} \times \mathcal{D}^\delta$ ,  $\mathcal{D}^\delta \times \mathcal{D}$ , and  $\mathcal{D}^\delta \times \mathcal{D}^\delta$ .*
- (b) *For  $1 \leq p \leq \infty$ , biparameter  $A_p$  is the intersection of four translates of biparameter dyadic  $A_p$ :*

$$A_p(\mathbb{R} \otimes \mathbb{R}) = A_p^{d,d}(\mathbb{R} \otimes \mathbb{R}) \cap A_p^{d,\delta}(\mathbb{R} \otimes \mathbb{R}) \cap A_p^{\delta,d}(\mathbb{R} \otimes \mathbb{R}) \cap A_p^{\delta,\delta}(\mathbb{R} \otimes \mathbb{R}).$$

- (c) *For  $1 \leq p \leq \infty$ , biparameter  $RH_p$  is the intersection of four translates of biparameter dyadic  $RH_p$ :*

$$RH_p(\mathbb{R} \otimes \mathbb{R}) = RH_p^{d,d}(\mathbb{R} \otimes \mathbb{R}) \cap RH_p^{d,\delta}(\mathbb{R} \otimes \mathbb{R}) \cap RH_p^{\delta,d}(\mathbb{R} \otimes \mathbb{R}) \cap RH_p^{\delta,\delta}(\mathbb{R} \otimes \mathbb{R}).$$

The constant  $A_p(\omega)$  depends only on  $A_p^{d,d}(\omega)$ ,  $A_p^{d,\delta}(\omega)$ ,  $A_p^{\delta,d}(\omega)$ , and  $A_p^{\delta,\delta}(\omega)$ , and vice versa, and similarly for the other classes.

In the case of  $k$  parameters, the analogous results hold using the intersection of  $2^k$  translates of the dyadic classes.

*Proof.* The proof is by iteration of the one-parameter argument. We sketch the case of  $A_p$  for  $1 < p < \infty$ . The other cases are similar. Take  $\omega \in A_p(\mathbb{R} \otimes \mathbb{R})$ . By our one-parameter result (Theorem 3.3), for almost every  $y$  we have  $\omega(\cdot, y) \in A_p(\mathbb{R}) = A_p^d(\mathbb{R}) \cap A_p^\delta(\mathbb{R})$ , and similarly in the second variable. Thus  $\omega \in A_p^{d,d}(\mathbb{R} \otimes \mathbb{R})$ , and similarly  $\omega \in A_p^{d,\delta}(\mathbb{R} \otimes \mathbb{R})$ ,  $\omega \in A_p^{\delta,d}(\mathbb{R} \otimes \mathbb{R})$ , and  $\omega \in A_p^{\delta,\delta}(\mathbb{R} \otimes \mathbb{R})$ . Conversely, if  $\omega$  is in the intersection of the four dyadic spaces, then for a.e.  $y$ ,  $\omega(\cdot, y)$  belongs to  $A_p^d(\mathbb{R}) \cap A_p^\delta(\mathbb{R}) = A_p(\mathbb{R})$ . Similarly, for a.e.  $x$ ,  $\omega(x, \cdot)$  belongs to  $A_p(\mathbb{R})$ . Therefore  $\omega \in A_p(\mathbb{R} \otimes \mathbb{R})$ . Moreover, the claimed dependence of the constants follows immediately from the one-parameter result.  $\square$

## 4 BMO and product BMO

We extend Mei's BMO result to the biparameter case. We begin by recalling some observations and background results; for details see [CF, Ste]. Next we give a proof of Mei's one-parameter result, still using Mei's key lemma but expressing BMO in terms of Carleson measures. Then we extend this proof to the multiparameter case.

**Proposition 4.1.** *Let  $\psi \in C_c^\infty(\mathbb{R})$  be a smooth function, supported in the interval  $[-1, 1]$ , such that  $\int \psi(t) dt = 0$ . For  $y > 0$  let  $\psi_y(t) := \frac{1}{y} \psi(\frac{t}{y})$ . For  $t \in \mathbb{R}$  and  $y > 0$  let  $I_{t,y} := [t - y, t + y]$ . Then*

- (i) *if  $(t, y) \in T(I_0)$  then  $I_{t,y} \subset 3I_0$ , where  $3I_0$  is the interval with the same midpoint as  $I_0$  and length  $|3I_0| = 3|I_0|$ ,*
- (ii)  $\text{supp } \psi_y \subset [-y, y]$ ,
- (iii)  $\text{supp } \psi_y(t - \cdot) \subset I_{t,y}$ ,
- (iv) *for  $I \in \mathcal{D}$  or  $I \in \mathcal{D}^\delta$ , and for the Haar function  $h_I$  on  $I$ , if  $I \cap I_{t,y} = \emptyset$  then  $h_I * \psi_y(t) = 0$ , and*
- (v) *if  $I_{t,y} \subset Q_l$  or  $I_{t,y} \subset Q_r$ , where  $Q_l$  and  $Q_r$  are the left and right halves respectively of an interval  $Q$ , then  $h_I * \psi_y(t) = 0$ .*

We omit the (elementary) proofs, except to note that part (v) holds since  $h_Q$  is constant on each of  $Q_l$  and  $Q_r$ ,  $\psi_y$  is supported in  $I_{t,y}$ , and  $\int_{\mathbb{R}} \psi = 0$ . Now we impose an additional condition (the Calderón–Torchinsky condition) on  $\psi$  as follows: there exists a constant  $C_\psi$  such that for any  $\xi \neq 0$ ,

$$\int_0^\infty \frac{|\widehat{\psi}(\xi t)|^2}{t} dt \leq C_\psi. \quad (4.1)$$

**Definition 4.2.** For  $f \in L^1_{\text{loc}}(\mathbb{R})$  and for each dyadic interval  $J$ , define the projection  $P_J$  of  $f$  by

$$P_J f(x) := \sum_{I \in \mathcal{D}, I \subset J} (f, h_I) h_I(x).$$

If  $f \in L^2(\mathbb{R})$ , then  $\|f\|_{L^2(\mathbb{R})}^2 = \sum_{I \in \mathcal{D}} (f, h_I)^2$ . If  $f \in L^2(\mathbb{R})$ , then the following standard Littlewood–Paley  $L^2$  estimate holds:

$$\iint_{(t,y) \in \mathbb{R} \otimes \mathbb{R}_+} |f * \psi_y(t)|^2 \frac{dt dy}{y} \leq C_\psi \|f\|_{L^2(\mathbb{R})}^2$$

$\psi \in C_c^\infty(\mathbb{R})$ ,  $\int \psi = 0$  and satisfies (4.1).

**Definition 4.3.** A locally integrable function  $f$  belongs to the *dyadic BMO space*  $\text{BMO}_d(\mathbb{R})$  if there is a constant  $C$  such that

$$\|f\|_{\text{BMO}_d(\mathbb{R})} := \sup_{I \in \mathcal{D}} \int_I |f(x) - f_I| dx < \infty, \quad (4.2)$$

where  $f_I := \int_I f$ .

It follows from the John–Nirenberg theorem [Gar, Theorem 2.1, p.230] that for each  $p > 1$ , the expression

$$\|f\|_{\text{BMO}_{d,p}(\mathbb{R})} := \sup_{I \in \mathcal{D}} \left( \int_I |f(x) - f_I|^p dx \right)^{1/p}$$

is comparable to  $\|f\|_{\text{BMO}_d(\mathbb{R})}$ .

We also have the following equivalent definition of  $\text{BMO}_d(\mathbb{R})$  in terms of dyadic Carleson measures.

**Definition 4.4.** A locally integrable function  $f$  belongs to the *dyadic BMO space*  $\text{BMO}_d(\mathbb{R})$  if there is a constant  $C$  such that for all dyadic intervals  $J$ ,

$$\sum_{I \in \mathcal{D}, I \subset J} (f, h_I)^2 \leq C|J|. \quad (4.3)$$

We note that if in Definition 4.4 we allow  $J$  to range over all intervals in  $\mathbb{R}$ , not only dyadic intervals in  $\mathbb{R}$ , we recover the same dyadic BMO space  $\text{BMO}_d(\mathbb{R})$ , with comparable norms. This observation follows from Proposition 4.5 below and the fact that the sum in inequality (4.3) is over only dyadic intervals  $I$ .

The equivalence of conditions (4.2) and (4.3) can be seen as follows. First, suppose  $f$  satisfies (4.2). Then for each dyadic interval  $J$ ,

$$\begin{aligned} \frac{1}{|J|} \sum_{I \in \mathcal{D}, I \subset J} (f, h_I)^2 &= \frac{1}{|J|} \sum_{I \in \mathcal{D}, I \subset J} ((f - f_J)\chi_J, h_I)^2 \\ &\leq \frac{1}{|J|} \sum_{I \in \mathcal{D}} ((f - f_J)\chi_J, h_I)^2 \\ &\leq \frac{1}{|J|} \|(f - f_J)\chi_J\|_{L^2(\mathbb{R})}^2 \\ &= \frac{1}{|J|} \int_J |f(x) - f_J|^2 dx \\ &\leq C \|f\|_{\text{BMO}_d(\mathbb{R})}^2, \end{aligned} \quad (4.4)$$

which shows that  $f$  satisfies condition (4.3). Here we use  $\chi_J$  to denote the characteristic function on  $J$ ; in the first inequality we use the fact that  $(C, h_I) = 0$  for any constant  $C$ ; and in the last inequality, we use the fact that the dyadic BMO norms  $\|\cdot\|_{\text{BMO}_d(\mathbb{R})}$  and  $\|\cdot\|_{\text{BMO}_{d,2}(\mathbb{R})}$  are equivalent.



Conversely, suppose  $f$  satisfies (4.3). Then for each dyadic interval  $J$ ,

$$\begin{aligned}
\frac{1}{|J|} \int_J |f(x) - f_J| dx &\leq \left( \frac{1}{|J|} \int_J |f(x) - f_J|^2 dx \right)^{1/2} \\
&\leq \left( \frac{1}{|J|} \int |f(x) - f_J|^2 \chi_J(x) dx \right)^{1/2} \\
&\leq \left( \frac{1}{|J|} \sum_{I \in \mathcal{D}} ((f - f_J) \chi_J, h_I)^2 \right)^{1/2} \\
&\leq \left( \frac{1}{|J|} \sum_{I \in \mathcal{D}, I \subset J} ((f - f_I), h_I)^2 \right)^{1/2} \\
&\leq C^{1/2},
\end{aligned} \tag{4.5}$$

where the constant  $C$  in the last inequality is from condition (4.3).

From the above estimates, we see that the smallest constant  $C$  in condition (4.3) is comparable to  $\|f\|_{\text{BMO}_d(\mathbb{R})}^2$ .

We define  $\text{BMO}_\delta(\mathbb{R})$  similarly, in terms of both averages and Carleson conditions, with respect to the collection  $\mathcal{D}^\delta$  of translated dyadic intervals. Here  $\delta \in \mathbb{R}$ , and  $\mathcal{D}^\delta$  is defined at the start of Section 2.

**Proposition 4.5.** *Let  $K$  be any interval in  $\mathbb{R}$ , dyadic or not. Then  $K$  is contained in the union of two adjacent dyadic intervals  $J_1$  and  $J_2$  of equal length, i.e.,  $K \subset J_1 \cup J_2$ , with*

$$\frac{|J_1|}{2} = \frac{|J_2|}{2} < |K| \leq |J_1| = |J_2|.$$

*Proof.* Let  $N$  be the unique integer such that  $2^{N-1} < |K| \leq 2^N$ . Let  $J_1$  be the dyadic interval of length  $2^N$  that contains the left (or right) endpoint of  $K$ . If  $K \subset J_1$ , we are done. If  $K \not\subset J_1$ , then the right (or left) endpoint of  $K$  lies in the dyadic interval  $J_2$  of length  $2^N$  immediately to the right of  $J_1$ .  $\square$

**Theorem 4.6.** *Suppose  $\delta \in \mathbb{R}$  is far from dyadic rationals, in the sense of condition (2.1). Then  $\text{BMO} = \text{BMO}_d \cap \text{BMO}_\delta$ . Moreover, we have*

$$\max\{\|f\|_{\text{BMO}_d(\mathbb{R})}, \|f\|_{\text{BMO}_\delta(\mathbb{R})}\} \leq \|f\|_{\text{BMO}(\mathbb{R})} \leq (C \cdot C(\delta))^{1/2} \max\{\|f\|_{\text{BMO}_d(\mathbb{R})}, \|f\|_{\text{BMO}_\delta(\mathbb{R})}\},$$

where  $C$  depends only on  $C_\psi$  in (4.1).

*Proof.* The inclusion  $\text{BMO} \subset \text{BMO}_d \cap \text{BMO}_\delta$  is an immediate consequence of the definition

$$\text{BMO} := \{f \in L_{\text{loc}}^1 : \|f\|_* := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty\} \tag{4.6}$$

in terms of averages  $f_Q := (1/|Q|) \int_Q f$ , since for  $\text{BMO}_d$  and  $\text{BMO}_\delta$  we are taking the supremum over fewer intervals than for  $\text{BMO}$ . Moreover, we have  $\max\{\|f\|_{\text{BMO}_d(\mathbb{R})}, \|f\|_{\text{BMO}_\delta(\mathbb{R})}\} \leq \|f\|_{\text{BMO}(\mathbb{R})}$ .

Now we prove the other inclusion. Our proof, which relies on the Carleson-measure characterization of BMO, is more complicated than the original proof in [Mei]. We choose to give this proof because it readily generalizes to the multiparameter case (see Theorem 4.7). Suppose  $f$  belongs to both  $\text{BMO}_d(\mathbb{R})$  and  $\text{BMO}_\delta(\mathbb{R})$ . Choose a  $\psi$  as in Proposition 4.1 and satisfying (4.1). We must show that there is a positive constant  $C$  independent of  $I_0$  such that the inequality

$$\iint_{T(I_0)} |f * \psi_y(t)|^2 \frac{dt dy}{y} \leq C|I_0|$$

holds for all intervals  $I_0$ , where the constant  $C$  is comparable to  $\|f\|_{\text{BMO}(\mathbb{R})}^2$ .

Fix an interval  $I_0 \subset \mathbb{R}$ . For each point  $(t, y)$  in  $T(I_0)$ , let  $I_{t,y} := (t - y, t + y)$  be the interval of length  $2y$  centered at  $t$ . By Proposition 2.2, for each point  $(t, y) \in T(I_0)$  we may choose an interval  $I_{t,y}^*$  such that  $I_{t,y} \subset I_{t,y}^*$ ,  $|I_{t,y}^*| \leq C(\delta)|I_{t,y}|$ , and either  $I_{t,y}^* \in \mathcal{D}$  or  $I_{t,y}^* \in \mathcal{D}^\delta$ . Let

$$\mathcal{F}_1 := \{(t, y) \in T(I_0) \mid I_{t,y}^* \in \mathcal{D}\}, \quad \mathcal{F}_2 := \{(t, y) \in T(I_0) \mid I_{t,y}^* \in \mathcal{D}^\delta\}.$$

So  $T(I_0) = \mathcal{F}_1 \cup \mathcal{F}_2$ , and  $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ . Now we have

$$\iint_{(t,y) \in T(I_0)} |f * \psi_y(t)|^2 \frac{dt dy}{y} = \underbrace{\iint_{(t,y) \in \mathcal{F}_1} |f * \psi_y(t)|^2 \frac{dt dy}{y}}_{(G_1)} + \underbrace{\iint_{(t,y) \in \mathcal{F}_2} |f * \psi_y(t)|^2 \frac{dt dy}{y}}_{(G_2)}.$$

It suffices to control the term  $(G_1)$  since the estimate for the term  $(G_2)$  is similar. Replacing  $f$  by its Haar expansion, we see that

$$f * \psi_y(t) = \sum_{I \in \mathcal{D}} (f, h_I) h_I * \psi_y(t) = \sum_{I \in \mathcal{D}, I \cap I_{t,y} \neq \emptyset} (f, h_I) h_I * \psi_y(t), \quad (4.7)$$

since by Proposition 4.1(iv),  $h_I * \psi_y(t)$  can only be nonzero if  $I \cap I_{t,y} \neq \emptyset$ .

For each  $(t, y) \in \mathcal{F}_1$ , we have  $I_{t,y} \subset 3I_0$ , by Proposition 4.1(i).

Fix  $(t, y) \in \mathcal{F}_1$ . We split the sum in equation (4.7) at the scale of  $2^N|3I_0|$ , where  $N > 0$  is a constant to be determined later but independent of  $f, t, y$  and  $I_0$ . Let  $k_0$  be the unique integer such that

$$2^{-k_0} \leq |3I_0| < 2^{-k_0+1}.$$

Now,

$$f * \psi_y(t) = \underbrace{\sum_{k=k_0-N-1}^{\infty} \sum_{I \in \mathcal{D}_k, I \cap I_{t,y} \neq \emptyset} (f, h_I) h_I * \psi_y(t)}_{g_{11}} + \underbrace{\sum_{k=-\infty}^{k_0-N-2} \sum_{I \in \mathcal{D}_k, I \cap I_{t,y} \neq \emptyset} (f, h_I) h_I * \psi_y(t)}_{g_{12}}.$$

**For the sum  $g_{12}$ :** we first show that each term in the sum  $g_{12}$  over large intervals is zero, if  $N$  is chosen appropriately. Let  $N$  be the unique integer such that

$$2^N \leq 2C(\delta) < 2^{N+1}. \quad (4.8)$$

(Note that  $N \geq 2$ , since  $d(\delta) < 1$  and so  $2C(\delta) = 4/d(\delta) > 2^2$ .) We will use the right-hand inequality for our estimate of  $g_{12}$ , and the left-hand inequality for  $g_{11}$ .

If the interval  $I$  appears in the sum  $g_{12}$ , we have

$$|I| \geq 2^{-k_0+N+2} > 2^{N+1}|3I_0| > 2C(\delta)|I_{t,y}| \geq 2|I_{t,y}^*|.$$

Since the intervals  $I$  and  $I_{t,y}^*$  both belong to the same dyadic grid  $\mathcal{D}$  and  $|I| > |I_{t,y}^*|$ , it follows that either  $I$  and  $I_{t,y}^*$  are disjoint or  $I \supsetneq I_{t,y}^*$ . If the former, then  $h_I * \psi_y(t) = 0$ . If the latter, then since  $I_{t,y} \subset I_{t,y}^* \subsetneq I$  we see that  $I_{t,y}$  is contained in either the left half of  $I$  or the right half of  $I$ , and so by Proposition 4.1(v),  $h_I * \psi_y(t) = 0$ . Thus the sum  $g_{12}$  is zero.

**For the sum  $g_{11}$ :** For each interval  $I$  that appears in  $g_{11}$ , we have  $|I| \leq 2^{-k_0+N+1} \leq 2^{N+1}|3I_0|$  and  $I \cap 3I_0 \neq \emptyset$ . It follows that each such interval  $I$  is contained in the interval  $2^{N+1}9I_0$  that has the same midpoint as  $I_0$  and length  $2^{N+1}|9I_0|$ . For brevity, let

$$J_0 := 2^{N+1}9I_0.$$

We reiterate that if  $I \cap I_{t,y} \neq \emptyset$  then  $I \subset J_0$ .

Then

$$\begin{aligned} g_{11} &:= \sum_{k=k_0-N-1}^{\infty} \sum_{I \in \mathcal{D}_k, I \cap I_{t,y} \neq \emptyset} (f, h_I) h_I * \psi_y(t) \\ &= \sum_{k=k_0-N-1}^{\infty} \sum_{I \in \mathcal{D}_k, I \subset J_0} (f, h_I) h_I * \psi_y(t) \\ &= \sum_{I \in \mathcal{D}, I \subset J_0} (f, h_I) h_I * \psi_y(t). \end{aligned}$$

The third equality holds because if  $I \subset J_0$ ,  $I \in \mathcal{D}_k$  and  $k < k_0 - N - 1$ , then  $h_I * \psi_y(t) = 0$  by Proposition 4.1 and the argument for  $g_{12}$  above.

As a consequence, and applying the Carleson condition (4.3) for  $f \in \text{BMO}_d(\mathbb{R})$  and the interval  $J_0$ , we see that

$$\begin{aligned} (G_1) &= \iint_{\mathcal{F}_1} |g_{11}|^2 \frac{dt dy}{y} \leq \iint_{\mathcal{F}_1} \left| \sum_{I \in \mathcal{D}, I \subset J_0} (f, h_I) h_I * \psi_y(t) \right|^2 \frac{dt dy}{y} \\ &= \iint_{\mathcal{F}_1} |P_{J_0} f * \psi_y(t)|^2 \frac{dt dy}{y} \leq \iint_{(t,y) \in \mathbb{R} \otimes \mathbb{R}_+} |P_{J_0} f * \psi_y(t)|^2 \frac{dt dy}{y} \\ &\leq C \|P_{J_0} f\|_{L^2(\mathbb{R})}^2 \\ &= C \sum_{I \in \mathcal{D}} (P_{J_0} f, h_I)^2 \\ &= C \sum_{I \in \mathcal{D}, I \subset J_0} (f, h_I)^2 \end{aligned}$$

$$\begin{aligned}
&\leq C|J_0| \|f\|_{\text{BMO}_d(\mathbb{R})}^2 \\
&\leq C2^{N+1}|I_0| \|f\|_{\text{BMO}_d(\mathbb{R})}^2 \\
&\leq C \cdot C(\delta)|I_0| \|f\|_{\text{BMO}_d(\mathbb{R})}^2,
\end{aligned}$$

where  $C$  depends only on  $C_\psi$  in (4.1).

It follows that

$$(G_1) \leq C \cdot C(\delta)|I_0| \|f\|_{\text{BMO}_d(\mathbb{R})}^2.$$

In the same way, we obtain the analogous estimate for  $(G_2)$ , with  $\|f\|_{\text{BMO}_d(\mathbb{R})}$  replaced by  $\|f\|_{\text{BMO}_\delta(\mathbb{R})}$ :

$$(G_2) \leq C \cdot C(\delta)|I_0| \|f\|_{\text{BMO}_\delta(\mathbb{R})}^2.$$

Therefore

$$\|f\|_{\text{BMO}(\mathbb{R})} \leq (C \cdot C(\delta))^{1/2} \max\{\|f\|_{\text{BMO}_d(\mathbb{R})}, \|f\|_{\text{BMO}_\delta(\mathbb{R})}\}, \quad (4.9)$$

as required.  $\square$

We now turn to the product case. For simplicity we discuss the case of two parameters.

A locally integrable function  $f$  on  $\mathbb{R} \otimes \mathbb{R}$  belongs to the product BMO space  $\text{BMO}(\mathbb{R} \otimes \mathbb{R})$  if there exists a positive constant  $C$  such that for every open set  $\Omega \subset \mathbb{R} \otimes \mathbb{R}$  with finite measure, the following inequality holds:

$$\iint_{T(\Omega)} |f * \psi_{y_1} \psi_{y_2}(t_1, t_2)|^2 \frac{dt_1 dy_1 dt_2 dy_2}{y_1 y_2} \leq C|\Omega|. \quad (4.10)$$

Here  $T(\Omega) := \{(t_1, y_1, t_2, y_2) \mid I_{t_1, y_1} \times I_{t_2, y_2} \subset \Omega\}$  is the Carleson tent on  $\Omega$ . Also  $\psi_{y_1} \psi_{y_2}(t_1, t_2) = y_1^{-1} y_2^{-1} \psi(t_1/y_1) \psi(t_2/y_2)$ , where  $\psi$  is a function of the kind described above in the one-parameter case. The smallest such  $C$  is comparable to  $\|f\|_{\text{BMO}(\mathbb{R} \otimes \mathbb{R})}^2$ .

Next we mention the following four types of dyadic product BMO spaces  $\text{BMO}_{d,d}$ ,  $\text{BMO}_{d,\delta}$ ,  $\text{BMO}_{\delta,d}$  and  $\text{BMO}_{\delta,\delta}$ . They differ only in which of the dyadic grids  $\mathcal{D}$  and  $\mathcal{D}^\delta$  is used in each variable. First, a locally integrable function  $f$  on  $\mathbb{R} \otimes \mathbb{R}$  belongs to  $\text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$  if and only if there exists a positive constant  $C$  such that for each open set  $\Omega \subset \mathbb{R} \otimes \mathbb{R}$  with finite measure, the following inequality

$$\sum_{R=I \times J \in \mathcal{D} \times \mathcal{D}, R \subset \Omega} (f, h_R)^2 \leq C|\Omega| \quad (4.11)$$

holds, where  $h_R = h_I \times h_J$ , and  $h_I$  and  $h_J$  are the Haar functions on the intervals  $I \in \mathcal{D}$  and  $J \in \mathcal{D}$ , respectively.

Next, a locally integrable function  $f$  on  $\mathbb{R} \otimes \mathbb{R}$  belongs to  $\text{BMO}_{d,\delta}(\mathbb{R} \otimes \mathbb{R})$  if and only if there exists a positive constant  $C$  such that for each open set  $\Omega \subset \mathbb{R} \otimes \mathbb{R}$  with finite measure, the following inequality

$$\sum_{R=I \times J \in \mathcal{D} \times \mathcal{D}^\delta, R \subset \Omega} (f, h_R)^2 \leq C|\Omega| \quad (4.12)$$

holds, where  $h_R = h_I \times h_J$ ,  $h_I$  and  $h_J$  are the Haar functions on the intervals  $I \in \mathcal{D}$  and  $J \in \mathcal{D}^\delta$ , respectively.

We define  $\text{BMO}_{\delta,d}(\mathbb{R} \otimes \mathbb{R})$  and  $\text{BMO}_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R})$  similarly.

For simplicity we state and prove our multiparameter result for two parameters. However the statement and proof go through for arbitrarily many parameters.

**Theorem 4.7.** *Suppose  $\delta \in \mathbb{R}$  is far from dyadic rationals:  $\delta$  satisfies condition (2.1). Then*

$$\text{BMO}(\mathbb{R} \otimes \mathbb{R}) = \text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R}) \cap \text{BMO}_{d,\delta}(\mathbb{R} \otimes \mathbb{R}) \cap \text{BMO}_{\delta,d}(\mathbb{R} \otimes \mathbb{R}) \cap \text{BMO}_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R}).$$

*Bounds for the constants are given in the proof below.*

*In the case of  $k$  parameters, the analogous result holds using the intersection of  $2^k$  translates of the dyadic classes.*

*Proof.* We first note that  $\text{BMO}(\mathbb{R} \otimes \mathbb{R}) \subset \text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R}) \cap \text{BMO}_{d,\delta}(\mathbb{R} \otimes \mathbb{R}) \cap \text{BMO}_{\delta,d}(\mathbb{R} \otimes \mathbb{R}) \cap \text{BMO}_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R})$ . This inclusion is not trivial in the multiparameter setting. A proof (for biparameter BMO) was given in the Ph.D. thesis [P] of J. Pipher, but the best proof of this result is in S. Treil's paper [T]. There he shows that  $H^1(\mathbb{R} \otimes \mathbb{R}) \supset H^1_{d,d}(\mathbb{R} \otimes \mathbb{R})$  via the characterization of these  $H^1$  spaces in terms of the square function and the fact that the multiparameter square function acts iteratively when viewed as a vector-valued operator. Using the fact that the dual of  $H^1(\mathbb{R} \otimes \mathbb{R})$  is  $\text{BMO}(\mathbb{R} \otimes \mathbb{R})$ , by [CF], and likewise the dual of  $H^1_{d,d}(\mathbb{R} \otimes \mathbb{R})$  is  $\text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$ , by [Ber], it follows that  $\text{BMO}(\mathbb{R} \otimes \mathbb{R}) \subset \text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$ .

The same argument shows that  $\text{BMO}(\mathbb{R} \otimes \mathbb{R})$  is contained in each of  $\text{BMO}_{d,\delta}(\mathbb{R} \otimes \mathbb{R})$ ,  $\text{BMO}_{\delta,d}(\mathbb{R} \otimes \mathbb{R})$ , and  $\text{BMO}_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R})$ .

Now we prove the other inclusion. Suppose  $f \in \text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R}) \cap \text{BMO}_{d,\delta}(\mathbb{R} \otimes \mathbb{R}) \cap \text{BMO}_{\delta,d}(\mathbb{R} \otimes \mathbb{R}) \cap \text{BMO}_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R})$ . We will show that  $f \in \text{BMO}(\mathbb{R} \otimes \mathbb{R})$ .

We must show that there is a positive constant  $C$  such that the inequality

$$\iint_{T(\Omega)} |f * \psi_{y_1} \psi_{y_2}(t_1, t_2)|^2 \frac{dt_1 dy_1 dt_2 dy_2}{y_1 y_2} \leq C |\Omega| \quad (4.13)$$

holds for all open sets  $\Omega$  with finite measure.

Fix such a set  $\Omega \subset \mathbb{R} \otimes \mathbb{R}$ . For each point  $(t_1, y_1, t_2, y_2)$  in  $T(\Omega)$ , by definition the two intervals  $I_{t_1, y_1}$  and  $I_{t_2, y_2}$  satisfy  $I_{t_1, y_1} \times I_{t_2, y_2} \subset \Omega$ . By Proposition 2.2, for such  $(t_1, y_1)$ , we may choose an interval  $I_{t_1, y_1}^*$  such that  $I_{t_1, y_1} \subset I_{t_1, y_1}^*$ ,  $|I_{t_1, y_1}^*| \leq C(\delta)|I_{t_1, y_1}|$ , and either  $I_{t_1, y_1}^* \in \mathcal{D}$  or  $I_{t_1, y_1}^* \in \mathcal{D}^\delta$ . Similarly, for such  $(t_2, y_2)$ , we may choose an interval  $I_{t_2, y_2}^*$  such that  $I_{t_2, y_2} \subset I_{t_2, y_2}^*$ ,  $|I_{t_2, y_2}^*| \leq C(\delta)|I_{t_2, y_2}|$ , and either  $I_{t_2, y_2}^* \in \mathcal{D}$  or  $I_{t_2, y_2}^* \in \mathcal{D}^\delta$ . Now, we let

$$\begin{aligned} \mathcal{F}_1 &:= \{(t_1, y_1, t_2, y_2) \in T(\Omega) \mid I_{t_1, y_1}^* \in \mathcal{D}, \quad I_{t_2, y_2}^* \in \mathcal{D}\}; \\ \mathcal{F}_2 &:= \{(t_1, y_1, t_2, y_2) \in T(\Omega) \mid I_{t_1, y_1}^* \in \mathcal{D}, \quad I_{t_2, y_2}^* \in \mathcal{D}^\delta\}; \\ \mathcal{F}_3 &:= \{(t_1, y_1, t_2, y_2) \in T(\Omega) \mid I_{t_1, y_1}^* \in \mathcal{D}^\delta, \quad I_{t_2, y_2}^* \in \mathcal{D}\}; \\ \mathcal{F}_4 &:= \{(t_1, y_1, t_2, y_2) \in T(\Omega) \mid I_{t_1, y_1}^* \in \mathcal{D}^\delta, \quad I_{t_2, y_2}^* \in \mathcal{D}^\delta\}. \end{aligned}$$

Then  $T(\Omega) = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$ , and the sets  $\mathcal{F}_i$  ( $i = 1, 2, 3, 4$ ) are pairwise disjoint. As a consequence,

$$\begin{aligned} & \iint_{T(\Omega)} |f * \psi_{y_1} \psi_{y_2}(t_1, t_2)|^2 \frac{dt_1 dy_1 dt_2 dy_2}{y_1 y_2} \\ &= \sum_{i=1}^4 \iint_{\mathcal{F}_i} |f * \psi_{y_1} \psi_{y_2}(t_1, t_2)|^2 \frac{dt_1 dy_1 dt_2 dy_2}{y_1 y_2} \\ &=: (G_1) + (G_2) + (G_3) + (G_4). \end{aligned}$$

We first estimate  $(G_1)$ . For every  $(t_1, y_1, t_2, y_2)$  in  $\mathcal{F}_1$ , we have  $I_{t_1, y_1}^* \in \mathcal{D}$  and  $I_{t_2, y_2}^* \in \mathcal{D}$ . Let  $\tilde{I}_{t_1, y_1}^*$  and  $\tilde{I}_{t_2, y_2}^*$  be the parents of  $I_{t_1, y_1}^*$  and  $I_{t_2, y_2}^*$ , respectively. Define

$$\tilde{\Omega}_1 := \bigcup_{(t_1, y_1, t_2, y_2) \in \mathcal{F}_1} \tilde{I}_{t_1, y_1}^* \times \tilde{I}_{t_2, y_2}^*.$$

Then, we can easily see that

$$\begin{aligned} |\tilde{\Omega}_1| &= \left| \bigcup_{(t_1, y_1, t_2, y_2) \in \mathcal{F}_1} \tilde{I}_{t_1, y_1}^* \times \tilde{I}_{t_2, y_2}^* \right| = 2^2 \left| \bigcup_{(t_1, y_1, t_2, y_2) \in \mathcal{F}_1} I_{t_1, y_1}^* \times I_{t_2, y_2}^* \right| \\ &\leq 2^2 C(\delta)^2 \left| \bigcup_{(t_1, y_1, t_2, y_2) \in \mathcal{F}_1} I_{t_1, y_1} \times I_{t_2, y_2} \right| \leq 2^2 C(\delta)^2 |\Omega|. \end{aligned}$$

Next, using the biparameter Haar expansion, we have

$$f * \psi_{y_1} \psi_{y_2}(t_1, t_2) = \sum_{R=I_1 \times I_2 \in \mathcal{D} \times \mathcal{D}} (f, h_R) h_{I_1} * \psi_{y_1}(t_1) h_{I_2} * \psi_{y_2}(t_2)$$

for every  $(t_1, y_1, t_2, y_2) \in \mathcal{F}_1$ . We now claim that: If  $I_1 \not\subseteq \tilde{I}_{t_1, y_1}^*$ , then  $h_{I_1} * \psi_{y_1}(t_1) = 0$ .

In fact, this claim follows from the analogous estimates in the one-parameter case; see the estimates of  $g_{12}$  in the proof of Theorem 4.6. More precisely, we explain it as follows. First, from the properties of  $h_{I_1}$  and  $\psi_{y_1}$ , we see that if  $I_1 \cap \tilde{I}_{t_1, y_1}^* = \emptyset$ , then  $h_{I_1} * \psi_{y_1}(t_1) = 0$ . Moreover, if  $I_1 \cap \tilde{I}_{t_1, y_1}^* \neq \emptyset$  and  $I_1 \not\subseteq \tilde{I}_{t_1, y_1}^*$ , then  $I_1$  must be larger than  $\tilde{I}_{t_1, y_1}^*$  since both  $I_1$  and  $\tilde{I}_{t_1, y_1}^*$  are dyadic, which means that  $I_1$  is some ancestor of  $\tilde{I}_{t_1, y_1}^*$ . In this case, since  $\psi_{y_1}(t_1 - \cdot)$  is supported in  $I_{t_1, y_1}$  and  $h_{I_1}$  is constant on  $I_{t_1, y_1}$ , we have  $h_{I_1} * \psi_{y_1}(t_1) = 0$ .

Combining the two cases, we see that the claim holds.

Similarly, if  $I_2 \not\subseteq \tilde{I}_{t_2, y_2}^*$ , then  $h_{I_2} * \psi_{y_2}(t_2) = 0$ .

As a consequence, we have

$$\begin{aligned} f * \psi_{y_1} \psi_{y_2}(t_1, t_2) &= \sum_{R=I_1 \times I_2 \in \mathcal{D} \times \mathcal{D}} (f, h_R) h_{I_1} * \psi_{y_1}(t_1) h_{I_2} * \psi_{y_2}(t_2) \\ &= \sum_{R=I_1 \times I_2 \in \mathcal{D} \times \mathcal{D}, R \subset \tilde{I}_{t_1, y_1}^* \times \tilde{I}_{t_2, y_2}^*} (f, h_R) h_{I_1} * \psi_{y_1}(t_1) h_{I_2} * \psi_{y_2}(t_2). \end{aligned}$$

We now estimate  $G_1$ . First let  $P_{\tilde{\Omega}_1} f$  denote the projection

$$P_{\tilde{\Omega}_1} f = \sum_{R=I_1 \times I_2 \in \tilde{\Omega}_1} (f, h_R) h_R.$$

From the results above, we have

$$\begin{aligned} (G_1) &:= \iint_{\mathcal{F}_1} |f * \psi_{y_1} \psi_{y_2}(t_1, t_2)|^2 \frac{dt_1 dy_1 dt_2 dy_2}{y_1 y_2} \\ &= \iint_{\mathcal{F}_1} \left| \sum_{R=I_1 \times I_2 \in \mathcal{D} \times \mathcal{D}, R \subset \tilde{I}_{t_1, y_1}^* \times \tilde{I}_{t_2, y_2}^*} (f, h_R) h_{I_1} * \psi_{y_1}(t_1) h_{I_2} * \psi_{y_2}(t_2) \right|^2 \frac{dt_1 dy_1 dt_2 dy_2}{y_1 y_2} \\ &= \iint_{\mathcal{F}_1} \left| \sum_{R=I_1 \times I_2 \in \mathcal{D} \times \mathcal{D}, R \in \tilde{\Omega}_1} (f, h_R) h_{I_1} * \psi_{y_1}(t_1) h_{I_2} * \psi_{y_2}(t_2) \right|^2 \frac{dt_1 dy_1 dt_2 dy_2}{y_1 y_2} \\ &= \iint_{\mathcal{F}_1} \left| P_{\tilde{\Omega}_1} f * \psi_{y_1}(t_1) \psi_{y_2}(t_2) \right|^2 \frac{dt_1 dy_1 dt_2 dy_2}{y_1 y_2}, \end{aligned}$$

Here the last equality holds since the terms  $R \in \tilde{\Omega}_1$  but  $R \not\subset \tilde{I}_{t_1, y_1}^* \times \tilde{I}_{t_2, y_2}^*$  are zero.

Then, using the  $L^2$  boundedness of the Littlewood–Paley  $g$ -function, we see that

$$\begin{aligned} G_1 &\leq \iint_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} \left| P_{\tilde{\Omega}_1} f * \psi_{y_1}(t_1) \psi_{y_2}(t_2) \right|^2 \frac{dt_1 dy_1 dt_2 dy_2}{y_1 y_2} \\ &\leq C \|P_{\tilde{\Omega}_1} f\|_{L^2(\mathbb{R} \otimes \mathbb{R})}^2 \\ &= C \sum_{R \in \mathcal{D} \times \mathcal{D}} (P_{\tilde{\Omega}_1} f, h_R)^2 \\ &= C \sum_{R \in \tilde{\Omega}_1} (f, h_R)^2 \\ &\leq C |\tilde{\Omega}_1| \|f\|_{\text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})}^2 \\ &\leq 4C \cdot C(\delta)^2 |\Omega| \|f\|_{\text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})}^2. \end{aligned}$$

Repeating the proof above, we find that  $(G_2) \leq 4C \cdot C(\delta)^2 |\Omega| \|f\|_{\text{BMO}_{d,\delta}(\mathbb{R} \otimes \mathbb{R})}^2$ ,  $(G_3) \leq 4C \cdot C(\delta)^2 |\Omega| \|f\|_{\text{BMO}_{\delta,d}(\mathbb{R} \otimes \mathbb{R})}^2$  and  $(G_4) \leq 4C \cdot C(\delta)^2 |\Omega| \|f\|_{\text{BMO}_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R})}^2$ . Combining the estimates from  $G_1$  to  $G_4$ , we see that inequality (4.13) holds with a constant  $C$  independent of  $\Omega$ , as required. In particular,

$$\begin{aligned} &\|f\|_{\text{BMO}(\mathbb{R} \otimes \mathbb{R})} \\ &\leq 2C^{\frac{1}{2}} C(\delta) \max\{\|f\|_{\text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})}, \|f\|_{\text{BMO}_{d,\delta}(\mathbb{R} \otimes \mathbb{R})}, \|f\|_{\text{BMO}_{\delta,d}(\mathbb{R} \otimes \mathbb{R})}, \|f\|_{\text{BMO}_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R})}\}. \quad \square \end{aligned}$$

## 5 VMO and product VMO

We begin with the one-parameter case.

The space VMO of functions of vanishing mean oscillation was introduced by Sarason in [Sar] as the set of integrable functions on the circle  $\mathbb{T}$  satisfying  $\lim_{\delta \rightarrow 0} \sup_{I: |I| \leq \delta} \int_I |f - f_I| dx = 0$ .

This space is the closure in the BMO norm of the subspace of  $\text{BMO}(\mathbb{T})$  consisting of all uniformly continuous functions on  $\mathbb{T}$ .

An analogous space  $\text{VMO}(\mathbb{R})$  on the real line was defined by Coifman and Weiss [CW], where they proved that it is the predual of the Hardy space  $H^1(\mathbb{R})$ .

**Definition 5.1** ([CW]).  $\text{VMO}(\mathbb{R})$  is the closure of the space  $C_0^\infty(\mathbb{R})$  in the  $\text{BMO}(\mathbb{R})$  norm (4.6).

An equivalent version of  $\text{VMO}(\mathbb{R})$  can be defined as follows.

**Definition 5.2.** The space  $\text{VMO}(\mathbb{R})$  is the set of all functions  $f \in \text{BMO}(\mathbb{R})$  satisfying the following conditions:

- (a)  $\lim_{\delta \rightarrow 0} \sup_{Q: |Q| < \delta} \int_Q |f - f_Q| dx = 0$ ;
- (b)  $\lim_{N \rightarrow \infty} \sup_{Q: |Q| > N} \int_Q |f - f_Q| dx = 0$ ; and
- (c)  $\lim_{R \rightarrow \infty} \sup_{Q: Q \cap B(0, R) = \emptyset} \int_Q |f - f_Q| dx = 0$ ,

where  $Q$  denotes an interval in  $\mathbb{R}$ .

For the proof of the equivalence of Definitions 5.1 and 5.2 of  $\text{VMO}(\mathbb{R})$ , see the Lemma in Section 3 of [U, pp.166–167]. See also [Bou, Theorem 7].

There is a third equivalent definition of  $\text{VMO}(\mathbb{R})$ , in terms of Carleson measures.

**Definition 5.3.** A function  $f \in \text{BMO}(\mathbb{R})$  belongs to  $\text{VMO}(\mathbb{R})$  if

- (a)  $\lim_{\delta \rightarrow 0} \sup_{Q: |Q| < \delta} \frac{1}{|Q|} \int_{T(Q)} |f * \psi_y(t)|^2 \frac{dt dy}{y} = 0$ ;
- (b)  $\lim_{N \rightarrow \infty} \sup_{Q: |Q| > N} \frac{1}{|Q|} \int_{T(Q)} |f * \psi_y(t)|^2 \frac{dt dy}{y} = 0$ ; and
- (c)  $\lim_{R \rightarrow \infty} \sup_{Q: Q \cap B(0, R) = \emptyset} \frac{1}{|Q|} \int_{T(Q)} |f * \psi_y(t)|^2 \frac{dt dy}{y} = 0$ ,

where  $\psi$  is any function of the form specified in Proposition 4.1.



The equivalence of Definitions 5.2 and 5.3 can be shown as follows. First, it is a routine estimate that

$$\frac{1}{|Q|} \int_{T(Q)} |\psi_y * f(t)|^2 \frac{dt dy}{y} \leq C \int_{4Q} |f(x) - f_{4Q}|^2 dx,$$

where  $Q$  is an arbitrary interval in  $\mathbb{R}$ ,  $4Q$  is the interval with the same midpoint as  $Q$  and four times the length, and  $C$  is a constant independent of  $Q$  and  $f$ . As a consequence, (a), (b) and (c) in Definition 5.3 follow directly from (a), (b) and (c) in Definition 5.2. Conversely, suppose  $f$  satisfies Definition 5.3. Then it follows from Proposition 3.3 in [DDSTY] that  $f$  satisfies Definition 5.2. We note that [DDSTY] deals with the generalized space  $\text{VMO}_L(\mathbb{R}^n)$  of VMO functions associated to a differential operator  $L$  satisfying the conditions that  $L$  has a bounded holomorphic functional calculus on  $L^2(\mathbb{R}^n)$  and that the heat kernel of the analytic semigroup generated by  $L$  has suitable upper bounds. We need only the special case when  $L$  is the Laplacian  $\Delta$ . It is shown in Proposition 3.6 in [DDSTY], by an argument using the tent space corresponding to VMO, that  $\text{VMO}_\Delta(\mathbb{R}^n)$  coincides with the usual VMO as in Definition 5.2.

We turn to the dyadic one-parameter case. Again we give three equivalent definitions. First,  $\text{VMO}_d(\mathbb{R})$  is the closure of the space  $C_0^\infty(\mathbb{R})$  in the dyadic  $\text{BMO}_d(\mathbb{R})$  norm of formula (4.2). Second, in terms of averages, we define  $\text{VMO}_d(\mathbb{R})$  as in Definition 5.2 of  $\text{VMO}(\mathbb{R})$  but taking the three suprema over only dyadic intervals  $I$  instead of arbitrary intervals  $Q$ . The third definition is in terms of a Carleson condition on Haar coefficients as follows.

**Definition 5.4.** A function  $f \in \text{BMO}_d(\mathbb{R})$  belongs to the dyadic VMO space  $\text{VMO}_d(\mathbb{R})$  if

- (a)  $\lim_{\delta \rightarrow 0} \sup_{J: J \in \mathcal{D}} \frac{1}{|J|} \sum_{I: I \subset J, I \in \mathcal{D}, |I| < \delta} (f, h_I)^2 = 0;$
- (b)  $\lim_{N \rightarrow \infty} \sup_{J: J \in \mathcal{D}} \frac{1}{|J|} \sum_{I: I \subset J, I \in \mathcal{D}, |I| > N} (f, h_I)^2 = 0; \text{ and}$
- (c)  $\lim_{R \rightarrow \infty} \sup_{J: J \in \mathcal{D}} \frac{1}{|J|} \sum_{I: I \subset J, I \in \mathcal{D}, I \cap B(0, R) = \emptyset} (f, h_I)^2 = 0.$

We note that, as for dyadic BMO (Definition 4.4), allowing  $J$  in Definition 5.4 to range over all intervals, not just dyadic intervals, produces the same space  $\text{VMO}_d(\mathbb{R})$ , with a comparable norm.

**Proposition 5.5.** *The following three definitions of the dyadic VMO space  $\text{VMO}_d(\mathbb{R})$  are equivalent.*

- (1) *The definition of  $\text{VMO}_d(\mathbb{R})$  as the closure of  $C_0^\infty(\mathbb{R})$  in the  $\text{BMO}_d(\mathbb{R})$  norm (4.2) in terms of average.*
- (2) *The dyadic version Definition 5.2, in terms of averages.*
- (3) *Definition 5.4, in terms of Haar coefficients.*

*Proof.* The proof of the equivalence of definitions (1) and (2) follows the corresponding proof in the continuous case. The equivalence of definitions (2) and (3) follows directly from the estimates (4.4) and (4.5) in our proof of the equivalence of Definitions 4.3 and 4.4 of  $\text{BMO}_d(\mathbb{R})$ .  $\square$

Similarly, for each  $\delta \in \mathbb{R}$ , there are three equivalent definitions for the dyadic VMO space  $\text{VMO}_\delta(\mathbb{R})$  with respect to the collection  $\mathcal{D}^\delta$  of translated dyadic intervals, which is defined at the start of Section 2.

Next we consider the product VMO space  $\text{VMO}(\mathbb{R} \otimes \mathbb{R})$ , for simplicity with only two parameters. Here we give only two equivalent definitions, since the one-parameter definition in terms of averages does not generalize naturally.

First,  $\text{VMO}(\mathbb{R} \otimes \mathbb{R})$  is the closure of  $C_0^\infty(\mathbb{R} \otimes \mathbb{R})$  in the product  $\text{BMO}(\mathbb{R} \otimes \mathbb{R})$  norm. The second definition is in terms of Carleson measures, as follows.

**Definition 5.6.** A function  $f \in \text{BMO}(\mathbb{R} \otimes \mathbb{R})$  belongs to  $\text{VMO}(\mathbb{R} \otimes \mathbb{R})$  if

- (a)  $\lim_{\delta \rightarrow 0} \sup_{\Omega} \frac{1}{|\Omega|} \sum_{R \in \mathcal{D} \times \mathcal{D}: R \subset \Omega, |R| < \delta} \int_{T(R)} |f * \psi_y(t)|^2 \frac{dt_1 dy_1 dt_2 dy_2}{y_1 y_2} = 0;$
- (b)  $\lim_{N \rightarrow \infty} \sup_{\Omega} \frac{1}{|\Omega|} \sum_{R \in \mathcal{D} \times \mathcal{D}: R \subset \Omega, |R| > N} \int_{T(R)} |f * \psi_y(t)|^2 \frac{dt_1 dy_1 dt_2 dy_2}{y_1 y_2} = 0;$  and
- (c)  $\lim_{R \rightarrow \infty} \sup_{\Omega} \frac{1}{|\Omega|} \sum_{R \in \mathcal{D} \times \mathcal{D}: R \subset \Omega, R \not\subset B(0, N)} \int_{T(R)} |f * \psi_y(t)|^2 \frac{dt_1 dy_1 dt_2 dy_2}{y_1 y_2} = 0.$

Here and in the definitions below,  $\Omega$  ranges over all open sets in  $\mathbb{R} \otimes \mathbb{R}$  of finite measure.

A short calculation shows that Definition 5.6 is equivalent to the definition of  $\text{VMO}(\mathbb{R} \otimes \mathbb{R})$  given in [LTW, Prop 5.1(ii)]. In [LTW] the equivalence of this last definition and the definition in terms of  $C_0^\infty(\mathbb{R} \otimes \mathbb{R})$  is proved.

Finally, we define the dyadic product VMO space  $\text{VMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$ , in two ways. First,  $\text{VMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$  is the closure of  $C_0^\infty(\mathbb{R} \otimes \mathbb{R})$  in the dyadic  $\text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$  norm. The second definition is in terms of a Carleson condition on the Haar coefficients, as follows.

**Definition 5.7.** A function  $f \in \text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$  belongs to the *dyadic product VMO space*  $\text{VMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$  if

- (a)  $\lim_{\delta \rightarrow 0} \sup_{\Omega} \frac{1}{|\Omega|} \sum_{R \in \mathcal{D} \times \mathcal{D}: R \subset \Omega, |R| < \delta} (f, h_R)^2 = 0;$
- (b)  $\lim_{N \rightarrow \infty} \sup_{\Omega} \frac{1}{|\Omega|} \sum_{R \in \mathcal{D} \times \mathcal{D}: R \subset \Omega, |R| > N} (f, h_R)^2 = 0;$  and
- (c)  $\lim_{N \rightarrow \infty} \sup_{\Omega} \frac{1}{|\Omega|} \sum_{R \in \mathcal{D} \times \mathcal{D}: R \subset \Omega, R \not\subset B(0, N)} (f, h_R)^2 = 0.$

We define  $\text{VMO}_{d,\delta}(\mathbb{R} \otimes \mathbb{R})$ ,  $\text{VMO}_{\delta,d}(\mathbb{R} \otimes \mathbb{R})$ , and  $\text{VMO}_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R})$  similarly.

**Proposition 5.8.** *The following two definitions of dyadic product  $\text{VMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$  are equivalent.*

- (1) *The definition of  $\text{VMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$  as the closure of  $C_0^\infty(\mathbb{R} \otimes \mathbb{R})$  in the  $\text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$  norm.*
- (2) *Definition 5.7, in terms of Haar coefficients.*

*The corresponding result holds for each of  $\text{VMO}_{d,\delta}(\mathbb{R} \otimes \mathbb{R})$ ,  $\text{VMO}_{\delta,d}(\mathbb{R} \otimes \mathbb{R})$ , and  $\text{VMO}_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R})$ .*

*Proof.* To prove this proposition, we follow the ideas given in [LTW] for the continuous case. Denote by FH the linear space of finite linear combinations of the Haar basis  $\{h_R : R \in \mathcal{D} \times \mathcal{D}\}$ .

We first claim that

$$\text{clos}_{\text{BMO}_{d,d}} \text{FH} = \text{VMO}_{d,d}.$$

In fact, from Definition 5.7, it is immediate that every Haar function  $h_R$  belongs to  $\text{VMO}_{d,d}$ . Conversely, for each  $f \in \text{VMO}_{d,d}$ , set

$$f_n := \sum_{R \in \mathcal{D} \times \mathcal{D}: R \subset B(0, 2^n), 2^{-n} \leq |R| \leq 2^n} (f, h_R) h_R \quad (5.1)$$

for every positive integer  $n$ . Then it is clear that  $f_n \in \text{FH}$  for each  $n$ . Moreover,  $\|f - f_n\|_{\text{BMO}_{d,d}}$  goes to 0 as  $n$  tends to infinity, by conditions (a), (b) and (c) in Definition 5.7. Hence the claim holds.

Next, we claim that

$$\text{clos}_{\text{BMO}_{d,d}} C_0^\infty = \text{clos}_{\text{BMO}_{d,d}} \text{FH}.$$

In fact, we can see that  $C_0^\infty \subset \text{clos}_{\text{BMO}_{d,d}} \text{FH}$  since for every  $\varphi \in C_0^\infty$ , we can verify that  $\varphi$  satisfies conditions (a), (b) and (c) in Definition 5.7. Hence by taking  $\varphi_n$  as in equation (5.1), we can approximate  $\varphi$  by functions in FH. Conversely, it is easy to verify that  $\text{FH} \subset \text{clos}_{\text{BMO}_{d,d}} C_0^\infty$ .

The proof of Proposition 5.8 is complete.  $\square$

**Theorem 5.9.** *Suppose  $\delta \in \mathbb{R}$  satisfies condition (2.1). Then in the one-parameter case,*

$$\text{VMO}(\mathbb{R}) = \text{VMO}_d(\mathbb{R}) \cap \text{VMO}_\delta(\mathbb{R}),$$

*and in the multiparameter case (stated for two parameters for simplicity),*

$$\text{VMO}(\mathbb{R} \otimes \mathbb{R}) = \text{VMO}_{d,d}(\mathbb{R} \otimes \mathbb{R}) \cap \text{VMO}_{d,\delta}(\mathbb{R} \otimes \mathbb{R}) \cap \text{VMO}_{\delta,d}(\mathbb{R} \otimes \mathbb{R}) \cap \text{VMO}_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R}).$$

*Proof.* We first prove the one-parameter case.

The inclusion  $\text{VMO}(\mathbb{R}) \subset \text{VMO}_d(\mathbb{R}) \cap \text{VMO}_\delta(\mathbb{R})$  follows directly from the definitions of  $\text{VMO}(\mathbb{R})$ ,  $\text{VMO}_d(\mathbb{R})$ , and  $\text{VMO}_\delta(\mathbb{R})$  via averaging.

The proof of the other inclusion  $\text{VMO}(\mathbb{R}) \supset \text{VMO}_d(\mathbb{R}) \cap \text{VMO}_\delta(\mathbb{R})$  involves only minor modifications of our proof for  $\text{BMO}(\mathbb{R})$  above. We use the definition of  $\text{VMO}_d(\mathbb{R})$  and  $\text{VMO}_\delta(\mathbb{R})$  in terms of Haar coefficients (Definition 5.4). The key point is that the constant  $C$  in inequality (4.9) is replaced by the  $\varepsilon$  from Definition 5.4. We omit the details.

For the case of product VMO, we first show that  $\text{VMO}(\mathbb{R} \otimes \mathbb{R}) \subset \text{VMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$ . Take  $f \in \text{VMO}(\mathbb{R} \otimes \mathbb{R})$ . Then  $f$  is the limit in the  $\text{BMO}(\mathbb{R} \otimes \mathbb{R})$  norm of a sequence of functions  $f_n$  in  $C_0^\infty(\mathbb{R} \otimes \mathbb{R})$ . Then  $\{f_n\} \subset \text{BMO}(\mathbb{R} \otimes \mathbb{R}) \subset \text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$ , and also  $f_n$  converges to  $f$  in the  $\text{BMO}_d(\mathbb{R} \otimes \mathbb{R})$  norm. Therefore  $f$  belongs to  $\text{VMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$ , as required. The same argument shows that  $f$  belongs to each of  $\text{VMO}_{d,\delta}(\mathbb{R} \otimes \mathbb{R})$ ,  $\text{VMO}_{\delta,d}(\mathbb{R} \otimes \mathbb{R})$ , and  $\text{VMO}_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R})$ .

Again, we can prove the other inclusion via minor modifications of our proof for  $\text{BMO}(\mathbb{R} \otimes \mathbb{R})$  above, using the definition of our dyadic product VMO spaces in terms of Haar coefficients (Definition 5.7). Again, we omit the details.  $\square$

## 6 Hardy spaces and maximal functions

We begin with the one-parameter case. Denote by  $H^1$  the classical Hardy space and denote by  $H_d^1$  (resp.  $H_\delta^1$ ) the dyadic Hardy space with respect to  $\mathcal{D}$  (resp.  $\mathcal{D}^\delta$ ). Also, denote by  $M(f)$  the classical Hardy–Littlewood maximal function, and denote by  $M_d(f)$  (resp.  $M_\delta(f)$ ) the dyadic Hardy–Littlewood maximal function with respect to  $\mathcal{D}$  (resp.  $\mathcal{D}^\delta$ ).

Then we have the following results.

**Proposition 6.1.** *Suppose  $\delta \in (0,1)$  is far from dyadic rationals:  $d(\delta) > 0$ . Then the following relations hold between the continuous and dyadic versions.*

- (i)  $H^1 = H_d^1 + H_\delta^1$  with equivalent norms.
- (ii) For each  $f \in L_{\text{loc}}^1$ ,  $M(f)$  is comparable with  $M_d(f) + M_\delta(f)$  pointwise, and the implicit constants are independent of  $f$ .

*Proof.* For part (i), see Corollary 2.4 of [Mei].

For part (ii), it is immediate from the definitions that for  $f \in L_{\text{loc}}^1$ ,  $M_d(f) \leq M(f)$  and  $M_\delta(f) \leq M(f)$ . Thus,  $M_d(f) + M_\delta(f) \leq 2M(f)$ . Next, for each interval  $Q \subset \mathbb{R}$ , by Proposition 2.2 there is a suitable interval  $I \in \mathcal{D}$  or  $I \in \mathcal{D}^\delta$  such that

$$\frac{1}{|Q|} \int_Q |f| \leq C(\delta) \frac{1}{|I|} \int_I |f|.$$

As a consequence, we have  $M(f) \leq C(\delta) \max\{M_d(f), M_\delta(f)\}$ .  $\square$

Now we give a corollary of Theorem 3.3 and Proposition 6.1, for weighted Hardy spaces and weighted maximal functions.

Suppose  $\omega$  is a doubling weight. Denote by  $H^1(\omega)$  the weighted Hardy space. Also, suppose  $\omega_d$  is a dyadic doubling weight (resp.  $\omega_\delta$  is a  $\delta$ -dyadic doubling weight), denote by  $H_d^1(\omega_d)$  (resp.  $H_\delta^1(\omega_\delta)$ ) the dyadic (resp.  $\delta$ -dyadic) Hardy space with respect to  $\mathcal{D}$  (resp.  $\mathcal{D}^\delta$ ).

Denote by  $M_\omega(f)$  the weighted Hardy–Littlewood maximal function:

$$M_\omega(f)(x) := \sup_{Q \ni x} \frac{1}{\omega(Q)} \int_Q |f(y)| \omega(y) dy,$$

where the supremum is taken over all intervals in  $\mathbb{R}$ . Similarly, for a dyadic doubling weight  $\omega_d$ , we define the dyadic weighted Hardy–Littlewood maximal function  $M_{d,\omega_d}(f)$ , and for a  $\delta$ -dyadic doubling weight  $\omega_\delta$ , we define the  $\delta$ -dyadic weighted Hardy–Littlewood maximal function  $M_{\delta,\omega_\delta}(f)$ ; here the supremum is taken over only the intervals  $I \in \mathcal{D}$  (resp.  $I \in \mathcal{D}^\delta$ ).

Then we have the following generalizations of Proposition 6.1 to the weighted case.

**Corollary 6.2.** *Suppose  $\delta \in (0, 1)$  is far from dyadic rationals:  $d(\delta) > 0$ . Suppose  $\omega$  is a doubling weight. Then the following relations hold.*

- (i)  $H^1(\omega) = H_d^1(\omega) + H_\delta^1(\omega)$  with equivalent norms.
- (ii) For each  $f \in L_{\text{loc}}^1$ ,  $M_\omega(f)$  is pointwise equivalent to  $M_{d,\omega}(f) + M_{\delta,\omega}(f)$ . Here the implicit constants are independent of  $f$ .

In particular, this corollary holds for  $w \in A_p$ ,  $1 \leq p \leq \infty$ .

*Proof.* We first prove (i). We recall the definition of atoms in weighted Hardy spaces. A function  $a$  is called an atom of the Hardy space  $H^1(\omega)$  if it satisfies

- (a)  $\text{supp } a \subset Q$  for some interval  $Q \subset \mathbb{R}$ ;
- (b)  $\|a\|_{L^2(\omega)} \leq \omega(Q)^{-1/2}$ ; and
- (c)  $\int a(x) \omega(x) dx = 0$ .

Similarly, the atoms of  $H_d^1(\omega)$  (resp.  $H_\delta^1(\omega)$ ) satisfies the same conditions (a), (b) and (c) above with the extra condition that  $Q \in \mathcal{D}$  (resp.  $Q \in \mathcal{D}^\delta$ ).

From the definitions of atoms, it is immediate that each  $H_d^1(\omega)$ -atom or  $H_\delta^1(\omega)$ -atom is an  $H^1(\omega)$ -atom. Thus,  $H_d^1(\omega) \subset H^1(\omega)$  and  $H_\delta^1(\omega) \subset H^1(\omega)$  with norms  $\|f\|_{H^1(\omega)} \leq \|f\|_{H_d^1(\omega)}$  and  $\|f\|_{H^1(\omega)} \leq \|f\|_{H_\delta^1(\omega)}$ .

We now prove the converse. Suppose  $a$  is an  $H^1(\omega)$ -atom satisfying (a), (b) and (c) with an interval  $Q$ . Then, by Proposition 2.2, there exists an interval  $I$  such that  $Q \subset I$ ,  $|I| \leq C(\delta)|Q|$  and  $I \in \mathcal{D}$  or  $I \in \mathcal{D}^\delta$ . Moreover, Proposition 3.4 implies that

$$\|a\|_{L^2(\omega)} \leq \omega(Q)^{-1/2} \leq C(\delta)^{1/2} (C_{\text{dy}})^{\frac{1}{2} \log_2(4C(\delta))} \omega(I)^{-1/2}.$$

Let  $C_0 := C(\delta)^{1/2} (C_{\text{dy}})^{\frac{1}{2} \log_2(4C(\delta))}$ . Then the above inequality implies that  $C_0^{-1}a$  is an  $H_d^1(\omega)$ -atom if  $I \in \mathcal{D}$ , and an  $H_\delta^1(\omega)$ -atom if  $I \in \mathcal{D}^\delta$ . Hence  $H_d^1(\omega) + H_\delta^1(\omega) \subset H^1(\omega)$  with norms  $\|f\|_{H_d^1(\omega)} + \|f\|_{H_\delta^1(\omega)} \leq C_0 \|f\|_{H^1(\omega)}$ .

Now we turn to (ii). From the definition of the weighted Hardy–Littlewood maximal functions, it is immediate to see that  $M_{d,\omega}(f) \leq M_\omega(f)$  and  $M_{\delta,\omega}(f) \leq M_\omega(f)$  for every

$f \in L^1_{\text{loc}}$ . Conversely, for each interval  $Q \subset \mathbb{R}$ , by Proposition 2.2, there exists an interval  $I$  such that  $Q \subset I$ ,  $|I| \leq C(\delta)|Q|$  and  $I \in \mathcal{D}$  or  $I \in \mathcal{D}^\delta$ . Moreover, from Proposition 3.4, we have

$$\omega(I) \leq (C_{\text{dy}})^{\log_2(4C(\delta))} \omega(Q) |I|/|Q| \leq C(\delta)(C_{\text{dy}})^{\log_2(4C(\delta))} \omega(Q).$$

Consequently, we obtain that

$$\frac{1}{\omega(Q)} \int_Q |f(y)| \omega(y) dy \leq C(\delta)(C_{\text{dy}})^{\log_2(4C(\delta))} \frac{1}{\omega(I)} \int_I |f(y)| \omega(y) dy,$$

which implies that  $M_\omega(f) \leq C(\delta)(C_{\text{dy}})^{\log_2(4C(\delta))} (M_{\delta,\omega}(f) + M_{d,\omega}(f))$ .  $\square$

Now we turn to the multiparameter case (stated for two parameters).

Denote by  $H^1(\mathbb{R} \otimes \mathbb{R})$  the product Hardy space. S.-Y. Chang and R. Fefferman [CF] showed that the dual of  $H^1(\mathbb{R} \otimes \mathbb{R})$  is the product BMO space  $\text{BMO}(\mathbb{R} \otimes \mathbb{R})$  as mentioned in Section 4. Recently, Lacey, Terwilleger and Wick showed that the predual of  $H^1(\mathbb{R} \otimes \mathbb{R})$  is the product VMO space  $\text{VMO}(\mathbb{R} \otimes \mathbb{R})$  as mentioned in Section 5.

Next, as mentioned in the proof of Theorem 4.7, we denote by  $H^1_{d,d}(\mathbb{R} \otimes \mathbb{R})$  the dyadic product Hardy space with respect to the dyadic rectangles  $R \in \mathcal{D} \times \mathcal{D}$ , whose norm is defined as

$$\|f\|_{H^1_{d,d}(\mathbb{R} \otimes \mathbb{R})} := \left\| \left( \sum_{R \in \mathcal{D} \times \mathcal{D}} (f, h_R)^2 |R|^{-1} \chi_R \right)^{1/2} \right\|_1,$$

where  $\chi_R$  is the characteristic function of  $R$ . For more information on the dyadic Hardy space, we refer to [T]. Similarly we can define the dyadic product Hardy spaces  $H^1_{d,\delta}(\mathbb{R} \otimes \mathbb{R})$ ,  $H^1_{\delta,d}(\mathbb{R} \otimes \mathbb{R})$  and  $H^1_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R})$ . It is known that the dual of  $H^1_{d,d}(\mathbb{R} \otimes \mathbb{R})$  is  $\text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$ . We point out that a direct proof can be found in [HLL, Thm 4.2] where they deal with a more general setting of product sequence spaces. Similarly the dual spaces of  $H^1_{d,\delta}(\mathbb{R} \otimes \mathbb{R})$ ,  $H^1_{\delta,d}(\mathbb{R} \otimes \mathbb{R})$  and  $H^1_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R})$  are the dyadic product BMO spaces  $\text{BMO}_{d,\delta}(\mathbb{R} \otimes \mathbb{R})$ ,  $\text{BMO}_{\delta,d}(\mathbb{R} \otimes \mathbb{R})$  and  $\text{BMO}_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R})$ , respectively.

Now we address the duality of  $\text{VMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$  with  $H^1_{d,d}(\mathbb{R} \otimes \mathbb{R})$ . In fact, the proof is similar to the proof of the continuous version  $((\text{VMO}(\mathbb{R} \otimes \mathbb{R}))^* = H^1(\mathbb{R} \otimes \mathbb{R}))$  as shown in [LTW], where they relied on the facts that  $(H^1(\mathbb{R} \otimes \mathbb{R}))^* = \text{BMO}(\mathbb{R} \otimes \mathbb{R})$  and  $\text{clos}_{H^1} \text{FW} = H^1$ . Here FW means the linear space of finite linear combinations of product wavelets. Correspondingly, we have the facts that  $(H^1_{d,d}(\mathbb{R} \otimes \mathbb{R}))^* = \text{BMO}_{d,d}(\mathbb{R} \otimes \mathbb{R})$  and that  $\text{clos}_{H^1_{d,d}} \text{FH} = H^1_{d,d}$ , where the latter follows from the definition of the norm of  $H^1_{d,d}(\mathbb{R} \otimes \mathbb{R})$ .

Thus, the equality  $(\text{VMO}_{d,d}(\mathbb{R} \otimes \mathbb{R}))^* = H^1_{d,d}(\mathbb{R} \otimes \mathbb{R})$  holds. Similar results hold for  $\text{VMO}_{d,\delta}(\mathbb{R} \otimes \mathbb{R})$ ,  $\text{VMO}_{\delta,d}(\mathbb{R} \otimes \mathbb{R})$  and  $\text{VMO}_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R})$ .

**Proposition 6.3.** *Suppose  $\delta \in \mathbb{R}$  satisfies condition (2.1). Then*

$$H^1(\mathbb{R} \otimes \mathbb{R}) = H^1_{d,d}(\mathbb{R} \otimes \mathbb{R}) + H^1_{d,\delta}(\mathbb{R} \otimes \mathbb{R}) + H^1_{\delta,d}(\mathbb{R} \otimes \mathbb{R}) + H^1_{\delta,\delta}(\mathbb{R} \otimes \mathbb{R}),$$

*with equivalent norms.*

This proposition follows from Theorem 5.9 by duality.

Next we turn to the maximal functions. Instead of the Hardy–Littlewood maximal function, in the multiparameter setting we consider the strong maximal function  $M_s$ , which is defined as follows. For a locally integrable function  $f \in \mathbb{R}^2$ ,

$$M_s(f)(x, y) := \sup_{R \ni (x, y)} \frac{1}{|R|} \int_R |f(u, v)| \, du \, dv, \quad (6.1)$$

where the supremum is taken over all rectangles  $R \in \mathbb{R}^2$ .

Next, we denote by  $M_s^{d,d}(f)$  the dyadic strong maximal function defined by restricting the supremum in formula (6.1) to dyadic rectangles  $R \in \mathcal{D} \times \mathcal{D}$ . Also denote by  $M_s^{d,\delta}(f)$  the dyadic strong maximal function where in (6.1) we take the supremum over all  $R \in \mathcal{D} \times \mathcal{D}^\delta$ . We define  $M_s^{\delta,d}(f)$  and  $M_s^{\delta,\delta}(f)$  similarly. Then we have the following result.

**Proposition 6.4.** *Suppose  $\delta \in \mathbb{R}$  satisfies condition (2.1). Then for each  $f \in L_{\text{loc}}^1(\mathbb{R}^2)$ ,  $M_s(f)$  is comparable with  $M_s^{d,d}(f) + M_s^{d,\delta}(f) + M_s^{\delta,d}(f) + M_s^{\delta,\delta}(f)$  pointwise, and the implicit constants are independent of  $f$ .*

*Proof.* The proof of this proposition is similar to that of Proposition 6.1(ii).  $\square$

In parallel with the one-parameter case, we define weighted product Hardy spaces  $H^1(\omega)$ ,  $H_{d,d}^1(\omega_{d,d})$ ,  $H_{d,\delta}^1(\omega_{d,\delta})$ ,  $H_{\delta,d}^1(\omega_{\delta,d})$ , and  $H_{\delta,\delta}^1(\omega_{\delta,\delta})$  for doubling weight  $\omega$  and dyadic doubling weights  $\omega_{d,d}$ ,  $\omega_{d,\delta}$ ,  $\omega_{\delta,d}$  and  $\omega_{\delta,\delta}$ . Also we have the weighted strong maximal functions  $M_{s,\omega}$ ,  $M_s^{d,d,\omega_{d,d}}$ ,  $M_s^{d,\delta,\omega_{d,\delta}}$ ,  $M_s^{\delta,d,\omega_{\delta,d}}$  and  $M_s^{\delta,\delta,\omega_{\delta,\delta}}$ .

Then, in parallel with Corollary 6.2, we have the weighted version of Propositions 6.3 and 6.4. We state it as follows, omitting the proof.

**Corollary 6.5.** *Suppose  $\delta \in (0, 1)$  is far from dyadic rationals:  $d(\delta) > 0$ . Suppose  $\omega$  is a product doubling weight. Then the following relations hold.*

- (i)  $H^1(\omega) = H_{d,d}^1(\omega) + H_{d,\delta}^1(\omega) + H_{\delta,d}^1(\omega) + H_{\delta,\delta}^1(\omega)$  with equivalent norms.
- (ii) For each  $f \in L_{\text{loc}}^1$ ,  $M_{s,\omega}$  is pointwise equivalent to  $M_s^{d,d,\omega}(f) + M_s^{d,\delta,\omega}(f) + M_s^{\delta,d,\omega}(f) + M_s^{\delta,\delta,\omega}(f)$ . Here the implicit constants are independent of  $f$ .

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